A higher dimensional fractional Borel-Pompeiu formula and a related hypercomplex fractional operator calculus*

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Abstract

In this paper we develop a fractional integro-differential operator calculus for Clifford-algebra valued functions. To do that we introduce fractional analogues of the Teodorescu and Cauchy-Bitsadze operators and we investigate some of their mapping properties. As a main result we prove a fractional Borel-Pompeiu formula based on a fractional Stokes formula. This tool in hand allows us to present a Hodge-type decomposition for the fractional Dirac operator. Our results exhibit an amazing duality relation between left and right operators and between Caputo and Riemann-Liouville fractional derivatives. We round off this paper by presenting a direct application to the resolution of boundary value problems related to Laplace operators of fractional order.

Keywords: Fractional Clifford analysis; Fractional derivatives; Stokes's formula; Borel-Pompeiu formula; Cauchy's integral formula; Hodge-type decomposition.

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1 Introduction

Clifford analysis offers a higher dimensional generalization of the classical theory of complex holomorphic functions. Its tools can be applied to several different areas, for instance to quantum mechanics, quantum field theory [15], projective geometry, computer graphics [30], neural network theory [3] and to many other areas of physics and engineering [17]. The corresponding analogy of the class of complex holomorphic functions is that of monogenic functions. These are the null solutions to the Dirac operator. The latter operator factorizes the Laplace operator and provides a first order generalization of the well-known Cauchy-Riemann operator in complex analysis (see [5, 8]).

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A main tool that Clifford holomorphic function theory uses in the treatment of boundary value problems is the Teodorescu operator, which is the right inverse of the Dirac operator. Properties and applications of the hypercomplex Teodorescu operator have been studied by many authors (see for instance [29] for a list of references). In the context of quaternionic and Clifford analysis, K. Gürlebeck and W. Sprößig studied among many others particular mapping and regularity properties of this integral operator. Furthermore, they studied its connections to elliptic boundary value problems (see [17]). Additionally, in [4] the authors also investigated some interesting connections between the Teodorescu operator and Hermitian regular functions. An extension to the time-dependent case addressing the heat and the Schrödinger operator has been presented subsequentially in [6].

Another central aspect that appears in the classical vector calculus and in generalized Clifford holomorphic function theories is the Helmholtz decomposition of L^2-spaces. Actually, in classical three-dimensional vector analysis it is nothing else than the decomposition of an arbitrary sufficiently regular vector field into the sum of a divergence free field (having a vector potential) and a curl free vector field (having a scalar potential). This particular space decomposition together with the Teodorescu operator calculus provides a very elegant resolution toolkit for boundary value problems in the corresponding scales of Hilbert-Sobolev spaces. For more details we also refer to the survey paper [27]. For the time-dependent case, see for instance [7, 22, 23].

A parallel development over the last years consists of a rapidly increasing interest in the theory of derivatives and integrals of non-integer order. Apart from several applications of fractional order models, as for example, to kinetic theories, statistical mechanics, to the dynamics in complex media, and to many other fields (see [28] and the references indicated therein), those methods provide an important counterpart and extension of the classical integer order models. The advantage of fractional models consists in the possibility of using fractional derivatives to describe the memory and hereditary properties of various materials and processes. Another field of application consists in addressing differential equations related to flows with permeable boundaries, such as for instance dam-fill problems which provides a further important motivation to develop three-dimensional generalizations of harmonic and Clifford analysis tools for the fractional setting. Preceding work pointing in this direction can be found in [18, 19] where the fractional p-Laplace equation has been treated.

Behind this background, the development of links between Clifford analysis and fractional differential calculus represents a very recent topic of research. In particular, some first steps in the direction of an introduction of a fractional Clifford analytic function theory have been made in [10–12, 14]. In these papers, the authors determined series representations for the fundamental solution related to some stationary and non-stationary fractional Dirac-type operators. The knowledge of explicit representation formulas of these fundamental solutions represents a corner stone in the development of a fractional version of Clifford analysis. The latter functions serve as kernels for fractional integral operators, such as the fractional Teodorescu operator that we are going to introduce and to investigate in this paper.

The aim of this paper is to apply the fundamental solutions obtained in [10, 12] in order to develop the fundamentals of a fractional operator calculus related to the fractional Dirac operator that depends on a vector of fractional parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in]0, 1]$, $i = 1, \ldots, n$. We introduce fractional analogues of the Teodorescu operator and of the Cauchy-Bitsadze operator, and we investigate some important mapping properties. Moreover, we present a Hodge-type decomposition for the fractional Dirac operator defined via left Caputo fractional derivatives. The results that we obtain exhibit an amazingly interesting "double duality" between left and right operators and between Caputo and Riemann-Liouville fractional derivatives. This double duality appears in a non-trivial generalization of the Stokes formula as well as in the fractional Borel-Pompeiu formula and in the Hodge-type decomposition that we are going to present subsequentially. Throughout the paper we show that we can always re-obtain the results of the classical function theory for the Dirac operator when switching to the limit case when $\alpha = (1, \ldots, 1)$. The analogous of the results presented in this paper for the case of the time-fractional parabolic Dirac operator can be found in [13].

The structure of the paper reads as follows. In the Preliminaries section we recall some basic definitions from the fractional calculus, special functions, and Clifford analysis. In Section 3, we present the fundamental solutions of the fractional Laplace and Dirac operators in $\mathbb{R}^n$, defined by left Riemann-Liouville and Caputo fractional derivatives. Moreover, we prove that these functions belong to the function space $L_1(\Omega)$ under certain conditions. Throughout the whole paper we assume that Ω is a bounded open rectangular domain. In Section 4, we introduce and study the main properties of the fractional analogues of the Teodorescu operator and of the

Cauchy-Bitsadze operator. Finally, in Section 5 we present a Hodge-type decomposition for the L_q-space, where one of the components is the kernel of the fractional Dirac operator defined in terms of left Caputo fractional derivatives. This decomposition represents a main result in the paper apart from proving the generalizations of the Borel-Pompeiu formulae in the context of Caputo derivatives. In the analysis of the mapping properties and the regularity properties there still appear some further peculiarities that require special attention. We round off this paper by giving an immediate application to the resolution of boundary value problems involving the fractional Laplace operators.

2 Preliminaries

2.1 Fractional calculus and special functions

Let $a, b \in \mathbb{R}$ with $a < b$ let $\alpha > 0$. The left and right Riemann-Liouville fractional integrals $I^\alpha_{a^+}$ and $I^\alpha_{b^-}$ of order α are given by (see [21])

$$\left(I^\alpha_{a^+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \quad x > a \tag{1}$$

$$\left(I^\alpha_{b^-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\, dt, \quad x < b. \tag{2}$$

By ${}^{RL}D^\alpha_{a^+}$ and ${}^{RL}D^\alpha_{b^-}$ we denote the left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ on $[a, b] \subset \mathbb{R}$, which are defined by (see [21])

$$\left({}^{RL}D^\alpha_{a^+} f\right)(x) = \left(D^m I^{m-\alpha}_{a^+} f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}}\, dt, \qquad x > a \tag{3}$$

$$\left({}^{RL}D^\alpha_{b^-} f\right)(x) = (-1)^m \left(D^m I^{m-\alpha}_{b^-} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b \frac{f(t)}{(t-x)^{\alpha-m+1}}\, dt, \qquad x < b. \tag{4}$$

Here, $m = [\alpha] + 1$ and $[\alpha]$ means the integer part of α. Let ${}^{C}D^\alpha_{a^+}$ and ${}^{C}D^\alpha_{b^-}$ denote, respectively, the left and right Caputo fractional derivative of order $\alpha > 0$ on $[a, b] \subset \mathbb{R}$, which are defined by (see [21])

$$\left({}^{C}D^\alpha_{a^+} f\right)(x) = \left(I^{m-\alpha}_{a^+} D^m f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}\, dt, \qquad x > a \tag{5}$$

$$\left({}^{C}D^\alpha_{b^-} f\right)(x) = (-1)^m \left(I^{m-\alpha}_{b^-} D^m f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b \frac{f^{(m)}(t)}{(t-x)^{\alpha-m+1}}\, dt, \qquad x < b. \tag{6}$$

We denote by $I^\alpha_{a^+}(L_1)$ the class of functions f that are represented by the fractional integral (1) of a summable function, that is $f = I^\alpha_{a^+}\varphi$, with $\varphi \in L_1(a, b)$. A description of this class of functions is given in [26].

Theorem 2.1 *(cf. [26]) A function f belongs to $I^\alpha_{a^+}(L_1)$, $\alpha > 0$, if and only if $I^{m-\alpha}_{a^+} f$ belongs to $AC^m([a, b])$, $m = [\alpha] + 1$ and $(I^{m-\alpha}_{a^+} f)^{(k)}(a) = 0$, $k = 0, \ldots, m-1$.*

In Theorem 2.1, $AC^m([a, b])$ denotes the class of functions f which are continuously differentiable on the segment $[a, b]$ up to the order $m-1$ and $f^{(m-1)}$ is supposed to be absolutely continuous on $[a, b]$. We note that the conditions $(I^{m-\alpha}_{a^+} f)^{(k)}(a) = 0$, $k = 0, \ldots, m-1$, imply that $f^{(k)}(a) = 0$, $k = 0, \ldots, m-1$ (see [25,26]). Removing the last condition in Theorem 2.1 we obtain the class of functions that admit a summable fractional derivative.

Definition 2.2 *(see [26]) A function $f \in L_1(a, b)$ has a summable fractional derivative $\left(D^\alpha_{a^+} f\right)(x)$ if $\left(I^{m-\alpha}_{a^+} f\right)(x)$ belongs to $AC^m([a, b])$, where $m = [\alpha] + 1$.*

If a function f admits a summable fractional derivative, then we have the following composition rules (see [26] and [25], respectively)

$$\left(I^\alpha_{a^+}\, {}^{RL}D^\alpha_{a^+} f\right)(x) = f(x) - \sum_{k=0}^{m-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \left(I^{m-\alpha}_{a^+} f\right)^{(m-k-1)}(a), \qquad m = [\alpha] + 1 \tag{7}$$

$$\left(I^\alpha_{a^+}\, {}^{C}D^\alpha_{a^+} f\right)(x) = f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{k!}(x-a)^k, \qquad m = [\alpha] + 1. \tag{8}$$

We remark that if $f \in I^{\alpha}_{a+}(L_1)$ then (7) and (8) reduce to $\left(I^{\alpha}_{a+}\,{}^{RL}D^{\alpha}_{a+} f\right)(x) = \left(I^{\alpha}_{a+}\,{}^{C}D^{\alpha}_{a+} f\right)(x) = f(x)$. Nevertheless we note that $D^{\alpha}_{a+}\ I^{\alpha}_{a+} f = f$ in both cases. This is a particular case of a more general property (cf. [25, (2.114)])

$$D^{\alpha}_{a+}\left(I^{\gamma}_{a+} f\right) = D^{\alpha-\gamma}_{a+} f, \qquad \alpha \geq \gamma > 0. \tag{9}$$

One important function used in this paper is the two-parameter Mittag-Leffler function $E_{\mu,\nu}(z)$ (see [16]), which is defined in terms of the power series by

$$E_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + \nu)}, \qquad \mu > 0,\ \nu > 0,\ z \in \mathbb{C}. \tag{10}$$

In particular, the function $E_{\mu,\nu}(z)$ is entire of order $\rho = \frac{1}{\mu}$ and type $\sigma = 1$. From the power series (10) and the operators (1), (3) and (5), we can obtain by straightforward calculations the following fractional integral and differential formulae involving $E_{\mu,\nu}(z)$ (see [16, pp. 87-88]):

$$I^{\alpha}_{a+}\left((x-a)^{\nu-1} E_{\mu,\nu}\left(k(x-a)^{\mu}\right)\right) = (x-a)^{\alpha+\nu-1} E_{\mu,\nu+\alpha}\left(k(x-a)^{\mu}\right) \tag{11}$$

for all $\alpha > 0, k \in \mathbb{C}, x > a, \mu > 0, \nu > 0$,

$${}^{RL}D^{\alpha}_{a+}\left((x-a)^{\nu-1} E_{\mu,\nu}\left(k(x-a)^{\mu}\right)\right) = (x-a)^{\nu-\alpha-1} E_{\mu,\nu-\alpha}\left(k(x-a)^{\mu}\right) \tag{12}$$

for all $\alpha > 0, k \in \mathbb{C}, x > a, \mu > 0, \nu > 0, \nu \neq \alpha - p$, where $p = 0, \ldots, m-1$ with $m = [\alpha] + 1$, and

$${}^{C}D^{\alpha}_{a+}\left((x-a)^{\nu-1} E_{\mu,\nu}\left(k(x-a)^{\mu}\right)\right) = (x-a)^{\nu-\alpha-1} E_{\mu,\nu-\alpha}\left(k(x-a)^{\mu}\right) \tag{13}$$

for all $\alpha > 0, k \in \mathbb{C}, x > a, \mu > 0, \nu > 0, \nu \neq p$, where $p = 1, \ldots, m$ with $m = [\alpha] + 1$.

Remark 2.3 *For $\nu = \alpha - p$ with $p = 0, \ldots, m-1$, we have that ${}^{RL}D^{\alpha}_{a+}((x-a)^{\alpha-p-1}) = 0$ which implies that the first term in the series expansion of $(x-a)^{\nu-1} E_{\mu,\nu}\left(k(x-a)^{\mu}\right)$ vanishes. Therefore, the derivation rule (12) must be replaced in these cases by the following derivation rule:*

$${}^{RL}D^{\alpha}_{a+}\left((x-a)^{\alpha-p-1} E_{\mu,\alpha-p}\left(k(x-a)^{\mu}\right)\right) = (x-a)^{\mu-p-1} k\, E_{\mu,\mu-p}\left(k(x-a)^{\mu}\right), \quad p = 0, \ldots, m-1. \tag{14}$$

Remark 2.4 *For $\nu = p$ with $p = 1, \ldots, m$, we have that ${}^{C}D^{\alpha}_{a+}((x-a)^{p-1}) = 0$ which implies that the first term in the series expansion of $(x-a)^{\nu-1} E_{\mu,\nu}\left(k(x-a)^{\mu}\right)$ vanishes. Therefore, the derivation rule (13) must be replaced in these cases by the following derivation rule:*

$${}^{C}D^{\alpha}_{a+}\left((x-a)^{p-1} E_{\mu,p}\left(k(x-a)^{\mu}\right)\right) = (x-a)^{\mu+p-\alpha-1} k\, E_{\mu,\mu+p-\alpha}\left(k(x-a)^{\mu}\right), \quad p = 1, \ldots, m. \tag{15}$$

The approach presented in this paper is based on the Laplace transform and leads to the solution of a linear Abel integral equation of the second kind.

Theorem 2.5 *([16, Thm. 4.2]) Let $f \in L_1[a,b], \alpha > 0$ and $\lambda \in \mathbb{C}$. Then the integral equation*

$$u(x) = f(x) + \frac{\lambda}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} u(t)\ dt, \qquad x \in [a,b]$$

has a unique solution

$$u(x) = f(x) + \lambda \int_a^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(\lambda(x-t)^{\alpha}) f(t)\ dt. \tag{16}$$

Now we recall the formula for fractional integration by parts for $0 < \alpha < 1$ and $x \in [a,b]$ (see [1])

$$\int_a^b g(x)\ \left({}^{C}D^{\alpha}_{a+} f\right)(x)\, dx \ = \int_a^b f(x)\ \left({}^{RL}D^{\alpha}_{b-} g\right)(x)\, dx + \left[f(x)\ \left(I^{\alpha}_{b-} g\right)(x)\right]_a^b,$$

$$\int_a^b g(x)\ \left({}^{C}D^{\alpha}_{b-} f\right)(x)\, dx \ = \int_a^b f(x)\ \left({}^{RL}D^{\alpha}_{a+} g\right)(x)\, dx - \left[f(x)\, I^{\alpha}_{a+} g(x)\right]_a^b.$$

We end this section by recalling an important result about the boundedness of the fractional integrals I^{α}_{a+} and I^{α}_{b-} (see Theorem 3.5 in [26]).

Theorem 2.6 *If $0 < \alpha < 1$ and $1 < p < \frac{1}{\alpha}$ then the operators I^{α}_{a+} and I^{α}_{b-} are bounded from $L_p(a,b)$ into $L_q(a,b)$, where $q = \frac{p}{1-\alpha p}$ and $[a,b] \subset \mathbb{R}$.*

2.2 Clifford analysis

Let $\{e_1, \cdots, e_n\}$ be the standard basis of the Euclidean vector space in $\mathbb{R}^n$. The associated Clifford algebra $\mathbb{R}_{0,n}$ is the free algebra generated by $\mathbb{R}^n$ modulo $x^2 = -||x||^2\, e_0$, where $x \in \mathbb{R}^n$ and e_0 is the neutral element with respect to the multiplication operation in the Clifford algebra $\mathbb{R}_{0,n}$. The defining relation induces the multiplication rules

$$e_i e_j + e_j e_i = -2\delta_{ij}, \tag{17}$$

where δ_{ij} denotes the Kronecker's delta. In particular, $e_i^2 = -1$ for all $i = 1, \dots, n$. The standard basis vectors thus operate as imaginary units. A vector space basis for $\mathbb{R}_{0,n}$ is given by the set $\{e_A : \; A \subseteq \{1, \dots, n\}\}$ with $e_A = e_{l_1} e_{l_2} \dots e_{l_r}$, where $1 \leq l_1 < \dots < l_r \leq n$, $0 \leq r \leq n$, $e_\emptyset := e_0 := 1$. Each $a \in \mathbb{R}_{0,n}$ can be written in the form $a = \sum_A a_A\, e_A$, with $a_A \in \mathbb{R}$. The conjugation in the Clifford algebra $\mathbb{R}_{0,n}$ is defined by $\overline{a} = \sum_A a_A\, \overline{e}_A$, where $\overline{e}_A = \overline{e}_{l_r}\, \overline{e}_{l_{r-1}} \dots \overline{e}_{l_1}$, and $\overline{e}_j = -e_j$ for $j = 1, \dots, n$, $\overline{e}_0 = e_0 = 1$. Each non-zero vector $a \in \mathbb{R}^n$ has a multiplicative inverse given by $\frac{\overline{a}}{||a||^2}$.

An $\mathbb{R}_{0,n}-$valued function f over $\Omega \subseteq \mathbb{R}^n$ has the representation $f = \sum_A e_A f_A$ with components $f_A : \Omega \to \mathbb{R}_{0,n}$. Properties such as continuity or differentiability have to be understood componentwise. Next, we recall the Euclidean Dirac operator $\mathcal{D} = \sum_{j=1}^n e_j \partial_{x_j}$. This operator satisfies $\mathcal{D}^2 = -\Delta$, where Δ is the n-dimensional Euclidean Laplacian. An $\mathbb{R}_{0,n}$-valued function f is called *left-monogenic* if it satisfies $\mathcal{D}u = 0$ on Ω (resp. *right-monogenic* if it satisfies $u\mathcal{D} = 0$ on Ω).

For more details about Clifford algebras and basic concepts of its associated function theory we refer the interested reader for example to [8].

3 Fundamental solutions revisited

In [10] and [12] the authors considered the so-called three-parameter fractional Laplace and Dirac operators defined in terms of the left Riemann-Liouville and Caputo fractional derivatives, and obtained families of eigenfunctions and fundamental solutions for both operators. In this section we present the generalization of these results for $\mathbb{R}^n$. Let $\Omega = \prod_{i=1}^n]a_i, b_i[$ be any bounded open rectangular domain, let $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_i \in]0, 1], i = 1, \dots, n$, and let us consider the n-parameter fractional Laplace operators ${}^{RL}\Delta^\alpha_{a^+}$ and ${}^{C}\Delta^\alpha_{a^+}$ defined over Ω by means of the left Riemann-Liouville and left Caputo fractional derivatives, respectively, given by

$${}^{RL}\Delta^\alpha_{a^+} = \sum_{i=1}^n {}^{RL}_{\;a_i^+}\partial^{1+\alpha_i}_{x_i}, \qquad {}^{C}\Delta^\alpha_{a^+} = \sum_{i=1}^n {}^{C}_{a_i^+}\partial^{1+\alpha_i}_{x_i}. \tag{18}$$

Associated to them there are the corresponding fractional Dirac operators ${}^{RL}\mathcal{D}^\alpha_{a^+}$ and ${}^{C}\mathcal{D}^\alpha_{a^+}$ defined by

$${}^{RL}\mathcal{D}^\alpha_{a^+} = \sum_{i=1}^n e_i\, {}^{RL}_{\;a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}, \qquad {}^{C}\mathcal{D}^\alpha_{a^+} = \sum_{i=1}^n e_i\, {}^{C}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}. \tag{19}$$

For $i = 1, \dots, n$ the partial derivatives ${}^{RL}_{\;a_i^+}\partial^{1+\alpha_i}_{x_i}$, ${}^{RL}_{\;a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}$, ${}^{C}_{a_i^+}\partial^{1+\alpha_i}_{x_i}$ and ${}^{C}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}$ are the left Riemann-Liouville and Caputo fractional derivatives (3) and (5) of orders $1 + \alpha_i$ and $\frac{1+\alpha_i}{2}$, with respect to the variable $x_i \in]a_i, b_i[$. Under certain conditions we have that ${}^{RL}\Delta^\alpha_{a^+} = -{}^{RL}\mathcal{D}^\alpha_{a^+}\, {}^{RL}\mathcal{D}^\alpha_{a^+}$ (see [10]), and ${}^{C}\Delta^\alpha_{a^+} = -\, {}^{C}\mathcal{D}^\alpha_{a^+}\, {}^{C}\mathcal{D}^\alpha_{a^+}$ (see [12]). Due to the nature of the eigenfunctions and the fundamental solution of these operators we additionally need to consider the variable $\widehat{x} = (x_2, \dots, x_n) \in \widehat{\Omega} = \prod_{i=2}^n]a_i, b_i[$, and the fractional Laplace and Dirac operators acting on $\widehat{x}$ defined by

$$\begin{aligned} {}^{RL}\widehat{\Delta}^\alpha_{a^+} &= \sum_{i=2}^n {}^{RL}_{\;a_i^+}\partial^{1+\alpha_i}_{x_i}, & {}^{C}\widehat{\Delta}^\alpha_{a^+} &= \sum_{i=2}^n {}^{C}_{a_i^+}\partial^{1+\alpha_i}_{x_i}, \\ {}^{RL}\widehat{\mathcal{D}}^\alpha_{a^+} &= \sum_{i=2}^n e_i\, {}^{RL}_{\;a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}, & {}^{C}\widehat{\mathcal{D}}^\alpha_{a^+} &= \sum_{i=2}^n e_i\, {}^{C}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}. \end{aligned} \tag{20}$$

We start by addressing the Caputo case. Consider the eigenfunction problem

$${}^{C}\Delta^\alpha_{a^+} v(x) = \lambda v(x), \tag{21}$$

where $\lambda \in \mathbb{C}$, and where we suppose that $v(x) = v(x_1, \dots, x_n)$ admits a summable fractional derivative $\left({}^{C}_{a_1^+}\partial^{1+\alpha_1}_{x_1} v\right)(x)$ in the variable x_1, and belongs to $I^{1+\alpha_i}_{a_i^+}(L_1)$ in the variables x_i, with $i = 2, \dots, n$. Applying the fractional integral $I^{1+\alpha_1}_{a_1^+}$ to both sides of the previous equation and taking into account (8) we get

$$v(x) - v(a_1, \widehat{x}) - (x_1 - a_1)\, v'_{x_1}(a_1, \widehat{x}) + \sum_{k=2}^{n} \left(I^{1+\alpha_1}_{a_1^+}\, {}^{C}_{a_k^+}\partial^{1+\alpha_k}_{x_k} v \right)(x) = \lambda \left(I^{1+\alpha_1}_{a_1^+} v \right)(x).$$

Now, applying successively the fractional integrals $I^{1+\alpha_j}_{a_j^+}$, with $j = 2, \dots, n$, to both sides of the previous equation, applying Fubini's theorem, and rearranging the terms, we get

$$\begin{aligned} &\left(I^{1+\alpha_1}_{a_1^+} \sum_{k=2}^{n} \prod_{\substack{j=2 \\ j\neq k}}^{n} I^{1+\alpha_j}_{a_j^+} v \right)(x) + \left(\prod_{j=2}^{n} I^{1+\alpha_j}_{a_j^+} v \right)(x) - \lambda \left(\prod_{j=1}^{n} I^{1+\alpha_j}_{a_j^+} v \right)(x) \\ &\qquad = \left(\prod_{j=2}^{n} I^{1+\alpha_j}_{a_j^+} g_0 \right)(\widehat{x}) + (x_1 - a_1) \left(\prod_{j=2}^{n} I^{1+\alpha_j}_{a_j^+} g_1 \right)(\widehat{x}), \end{aligned} \tag{22}$$

where g_0 and g_1 are the Cauchy initial conditions given by

$$g_0(\widehat{x}) = v(a_1, \widehat{x}) \qquad \text{and} \qquad g_1(\widehat{x}) = v'_{x_1}(a_1, \widehat{x}). \tag{23}$$

We observe that the fractional integrals in (22) are Laplace-transformable functions. Therefore, we may apply the $(n-1)$-dimensional Laplace transform with respect to $x_2, \dots, x_n$:

$$\mathfrak{F}(\widehat{s}) = \mathfrak{F}(s_2, \dots, s_n) = \mathfrak{L}\{f\}(s_2, \dots, s_n) = \int_{a_2}^{+\infty} \cdots \int_{a_n}^{+\infty} \exp\left(- \sum_{p=2}^{n} s_p x_p \right) f(x_2, \dots, x_n)\, dx_n \cdots dx_2.$$

Taking into account its convolution and operational properties (see [9,21]), we obtain the following relations for each term in (22):

$$\mathfrak{L}\left\{ I^{1+\alpha_1}_{a_1^+} \sum_{k=2}^{n} \prod_{\substack{j=2 \\ j\neq k}}^{n} I^{1+\alpha_j}_{a_j^+} v \right\}(x_1, \widehat{s}) = \sum_{k=2}^{n} \prod_{\substack{p=2 \\ p\neq k}}^{n} s_p^{-1-\alpha_p} \left(I^{1+\alpha_1}_{a_1^+} \mathcal{V} \right)(x_1, \widehat{s}), \qquad k = 2, \dots, n, \tag{24}$$

$$\mathfrak{L}\left\{ \prod_{j=2}^{n} I^{1+\alpha_j}_{a_j^+} v \right\}(x_1, \widehat{s}) = \prod_{p=2}^{n} s_p^{-1-\alpha_p} \mathcal{V}(x_1, \widehat{s}), \tag{25}$$

$$\mathfrak{L}\left\{ \prod_{j=1}^{n} I^{1+\alpha_j}_{a_j^+} v \right\}(x_1, \widehat{s}) = \prod_{p=2}^{n} s_p^{-1-\alpha_p} \left(I^{1+\alpha_1}_{a_1^+} \mathcal{V} \right)(x_1, \widehat{s}), \tag{26}$$

$$\mathfrak{L}\left\{ \prod_{j=2}^{n} I^{1+\alpha_j}_{a_j^+} g_0 \right\}(x_1, \widehat{s}) = \prod_{p=2}^{n} s_p^{-1-\alpha_p} \mathfrak{G}_0(\widehat{s}), \tag{27}$$

$$\mathfrak{L}\left\{ (x_1 - a_1) \left(\prod_{j=2}^{n} I^{1+\alpha_j}_{a_j^+} g_1 \right) \right\}(x_1, \widehat{s}) = (x_1 - a_1) \prod_{p=2}^{n} s_p^{-1-\alpha_p} \mathfrak{G}_1(\widehat{s}). \tag{28}$$

Combining all the resulting terms and multiplying by $\prod_{p=2}^{n} s_p^{1+\alpha_p}$ we obtain the following second kind homogeneous integral equation of Volterra type:

$$\mathcal{V}(x_1, \widehat{s}) + \frac{\sum_{p=2}^{n} s_p^{1+\alpha_p} - \lambda}{\Gamma(1+\alpha_1)} \int_{a_1}^{x_1} (x_1 - t)^{\alpha_1}\, \mathcal{V}(t, \widehat{s})\, dt = G(x_1, \widehat{s}), \tag{29}$$

where $G(x_1,\widehat{s}) = \mathfrak{G}_0(\widehat{s}) + (x_1-a_1)\,\mathfrak{G}_1(\widehat{s})$ and $\mathfrak{G}_k(\widehat{s}) = \mathfrak{L}\{g_k\}(s)$ with $k=0,1$. Using (16), we have that the unique solution of (29) in the class of summable functions is:

$$\mathcal{V}(x_1,\widehat{s}) = G(x_1,\widehat{s}) - \frac{\sum_{p=2}^{n} s_p^{1+\alpha_p} - \lambda}{\Gamma(1+\alpha_1)} \int_{a_1}^{x_1} (x_1-t)^{\alpha_1} E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-t)^{1+\alpha_1}\left(\sum_{p=2}^{n} s_p^{1+\alpha_p} - \lambda\right)\right) G(t,\widehat{s})\,dt, \tag{30}$$

which involves the two-parameter Mittag-Leffler function. Due the convergence of the integrals and the series that appear in (30), we can interchange them and rewrite (30) in the following way:

$$\begin{aligned}
\mathcal{V}(x_1,\widehat{s}) &= \left(1 + \sum_{k=0}^{+\infty}(-1)^{k+1} \frac{\left(\sum_{p=2}^{n} s_p^{1+\alpha_p} - \lambda\right)^{k+1}}{\Gamma((1+\alpha_1)k+2+\alpha_1)} (x_1-a_1)^{(1+\alpha_1)k+1+\alpha_1}\right)\mathfrak{G}_0(\widehat{s}) \\
&\quad + \left((x_1-a_1) + \sum_{k=0}^{+\infty}(-1)^{k+1} \frac{\left(\sum_{p=2}^{n} s_p^{1+\alpha_p} - \lambda\right)^{k+1}}{\Gamma((1+\alpha_1)(k+1)+2)} (x_1-a_1)^{(1+\alpha_1)(k+1)+1}\right)\mathfrak{G}_1(\widehat{s}) \\
&= \left(1 + \sum_{m=1}^{+\infty}(-1)^{m} \frac{\left(\sum_{p=2}^{n} s_p^{1+\alpha_p} - \lambda\right)^{m}}{\Gamma((1+\alpha_1)m+1)} (x_1-a_1)^{(1+\alpha_1)m}\right)\mathfrak{G}_0(\widehat{s}) \\
&\quad + \left((x_1-a_1) + \sum_{m=1}^{+\infty}(-1)^{m} \frac{\left(\sum_{p=2}^{n} s_p^{1+\alpha_p} - \lambda\right)^{m}}{\Gamma((1+\alpha_1)m+2)} (x_1-a_1)^{(1+\alpha_1)m+1}\right)\mathfrak{G}_1(\widehat{s}) \\
&= \sum_{m=0}^{+\infty}(-1)^{m} \frac{\left(\sum_{p=2}^{n} s_p^{1+\alpha_p} - \lambda\right)^{m}}{\Gamma((1+\alpha_1)m+1)} (x_1-a_1)^{(1+\alpha_1)m}\,\mathfrak{G}_0(\widehat{s}) \\
&\quad + \sum_{m=0}^{+\infty}(-1)^{m} \frac{\left(\sum_{p=2}^{n} s_p^{1+\alpha_p} - \lambda\right)^{m}}{\Gamma((1+\alpha_1)m+2)} (x_1-a_1)^{(1+\alpha_1)m+1}\,\mathfrak{G}_1(\widehat{s}).
\end{aligned} \tag{31}$$

In order to cancel the Laplace transform, we need to take into account its distributional form in Zemanian's space (for more details about generalized integral transforms see [31]) and the following relation:

$$\lim_{r_2,\ldots,r_n\to+\infty} \int_{\sigma_1-ir_2}^{\sigma_1+ir_2} \cdots \int_{\sigma_n-ir_n}^{\sigma_n+ir_n} \prod_{p=2}^{n} s_p^{n(1+\alpha_p)}\, G_k(\widehat{s}) \exp\left(\sum_{p=2}^{n} s_p x_p\right)\, ds_n \ldots ds_2 = \left(\prod_{j=2}^{n} \partial_{x_j^+}^{n(1+\alpha_j)}\, g_k\right)(\widehat{x}), \tag{32}$$

where $k=0,1$. Therefore, applying the multinomial theorem and after straightforward calculations we get the following solution of (21):

$$\begin{aligned}
v(x) &= \sum_{m=0}^{\infty}(-1)^m \frac{(x_1-a_1)^{(1+\alpha_1)m}}{\Gamma((1+\alpha_1)m+1)} \left({}^{C}\widehat{\Delta}_{a^+}^{\alpha} - \lambda\right)^m g_0(\widehat{x}) \\
&\quad + \sum_{m=0}^{\infty}(-1)^m \frac{(x_1-a_1)^{(1+\alpha_1)m+1}}{\Gamma((1+\alpha_1)m+2)} \left({}^{C}\widehat{\Delta}_{a^+}^{\alpha} - \lambda\right)^m g_1(\widehat{x}).
\end{aligned} \tag{33}$$

From the previous calculations we obtain the following results in $\mathbb{R}^n$, which generalize Theorem 3.1 and Theorem 4.1 in [12].

Theorem 3.1 *A family of eigenfunctions of the fractional Laplace operator ${}^{C}\Delta_{a^+}^{\alpha}$ is given by*

$$\begin{aligned}
v_\lambda(x) &= E_{1+\alpha_1,1}\left(-(x_1-a_1)^{1+\alpha_1}\left({}^{C}\widehat{\Delta}_{a^+}^{\alpha} - \lambda\right)\right) g_0(\widehat{x}) \\
&\quad + (x_1-a_1)\; E_{1+\alpha_1,2}\left(-(x_1-a_1)^{1+\alpha_1}\left({}^{C}\widehat{\Delta}_{a^+}^{\alpha} - \lambda\right)\right) g_1(\widehat{x}),
\end{aligned} \tag{34}$$

where $g_0(\widehat{x}) = v(a_1,\widehat{x})$ and $g_1(\widehat{x}) = v'_{x_1}(a_1,\widehat{x})$.

We give a direct proof of the theorem in order to confirm that (34) is indeed the solution of (21). The proof uses the fact that ${}^{C}_{a_1^+}\partial^{1+\alpha_1}_{x_1} 1 = 0$ and ${}^{C}_{a_1^+}\partial^{1+\alpha_1}_{x_1} (x_1 - a_1) = 0$, and the analogous fractional formula for differentiation of integrals depending on a parameter where the upper limit also depends on the same parameter (see [25, Section 2.7.4]). **Proof:** Applying the operator ${}^{C}\Delta^{\alpha}_{a^+} = {}^{C}_{a_1^+}\partial^{1+\alpha_1}_{x_1} + {}^{C}\widehat{\Delta}^{\alpha}_{a^+}$ to (34) and using the series expansion of the Mittag-Leffler function (10), we get

$$
\begin{aligned}
{}^{C}\Delta^{\alpha}_{a^+} v_\lambda(x) = & \sum_{k=1}^{\infty} (-1)^k \, \frac{(x_1 - a_1)^{(k-1)(1+\alpha_1)}}{\Gamma((1+\alpha_1)k - \alpha_1)} \, \left({}^{C}\widehat{\Delta}^{\alpha}_{a^+} - \lambda \right)^k g_0(\widehat{x}) \\
& + \sum_{k=1}^{\infty} (-1)^k \, \frac{(x_1 - a_1)^{(1+\alpha_1)k - \alpha_1}}{\Gamma((1+\alpha_1)k + 1 - \alpha_1)} \, \left({}^{C}\widehat{\Delta}^{\alpha}_{a^+} - \lambda \right)^k g_1(\widehat{x}) \quad + \quad \left({}^{C}\widehat{\Delta}^{\alpha}_{a^+} \right) v_\lambda(x).
\end{aligned}
$$

Rearranging the terms of the series we obtain

$$
{}^{C}\Delta^{\alpha}_{a^+} v_\lambda(x) \; = - \left({}^{C}\widehat{\Delta}^{\alpha}_{a^+} - \lambda \right) v_\lambda(x) + \left({}^{C}\widehat{\Delta}^{\alpha}_{a^+} \right) v_\lambda(x) \; = \lambda \, v_\lambda(x).
$$

■

Corollary 3.2 *A family of fundamental solutions for the fractional Laplace operator ${}^{C}\Delta^{\alpha}_{a^+}$ is obtained by considering $\lambda = 0$ in (34):*

$$
\begin{aligned}
{}^{C}G^{\alpha}_{+}(x) &= v_0(x) \\
&= E_{1+\alpha_1,1}\left(-(x_1 - a_1)^{1+\alpha_1} \;\; {}^{C}\widehat{\Delta}^{\alpha}_{a^+} \right) g_0(\widehat{x}) + (x_1 - a_1) \; E_{1+\alpha_1,2}\left(-(x_1 - a_1)^{1+\alpha_1} \;\; {}^{C}\widehat{\Delta}^{\alpha}_{a^+} \right) g_1(\widehat{x}), \quad (35)
\end{aligned}
$$

where $g_0(\widehat{x}) = v(a_1, \widehat{x})$ and $g_1(\widehat{x}) = v'_{x_1}(a_1, \widehat{x})$.

For the fractional Dirac operator ${}^{C}\mathcal{D}^{\alpha}_{a^+}$ we can obtain a family of fundamental solutions by applying the operator ${}^{C}\mathcal{D}^{\alpha}_{a^+}$ to the family of fundamental solutions of the operator ${}^{C}\Delta^{\alpha}_{a^+}$.

Theorem 3.3 *A family of fundamental solutions of the fractional Dirac operator ${}^{C}\mathcal{D}^{\alpha}_{a^+}$ (acting on the left or on the right) is given by*

$$
{}^{C}\mathcal{G}^{\alpha}_{+}(x) = \sum_{i=1}^{n} e_i \, \left({}^{C}\mathcal{G}^{\alpha}_{+} \right)_i (x), \tag{36}
$$

where

$$
\begin{aligned}
\left({}^{C}\mathcal{G}^{\alpha}_{+} \right)_1 (x) = & \, (x_1 - a_1)^{-\frac{1+\alpha_1}{2}} E_{1+\alpha_1, \frac{1-\alpha_1}{2}} \left(-(x_1 - a_1)^{1+\alpha_1} \;\; {}^{C}\widehat{\Delta}^{\alpha}_{a^+} \right) g_0(\widehat{x}) \\
& + (x_1 - a_1)^{\frac{1-\alpha_1}{2}} E_{1+\alpha_1, \frac{3-\alpha_1}{2}} \left(-(x_1 - a_1)^{1+\alpha_1} \;\; {}^{C}\widehat{\Delta}^{\alpha}_{a^+} \right) g_1(\widehat{x}),
\end{aligned} \tag{37}
$$

and for $i = 2, \ldots, n$

$$
\begin{aligned}
\left({}^{C}\mathcal{G}^{\alpha}_{+} \right)_i (x) = & \left(E_{1+\alpha_1,1} \left(-(x_1 - a_1)^{1+\alpha_1} \;\; {}^{C}\widehat{\Delta}^{\alpha}_{a^+} \right) \, {}^{C}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i} \right) g_0(\widehat{x}) \\
& + (x_1 - a_1) \left(E_{1+\alpha_1,2} \left(-(x_1 - a_1)^{1+\alpha_1} \;\; {}^{C}\widehat{\Delta}^{\alpha}_{a^+} \right) \, {}^{C}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i} \right) g_1(\widehat{x})
\end{aligned} \tag{38}
$$

with $g_0(\widehat{x}) = v(a_1, \widehat{x})$ and $g_1(\widehat{x}) = v'_{x_1}(a_1, \widehat{x})$.

Now we present the corresponding results for the Riemann-Liouville case. First we obtain the eigenfunctions associated to the operator ${}^{RL}\Delta^{\alpha}_{a^+}$ satisfying ${}^{RL}\Delta^{\alpha}_{a^+} v(x) = \lambda v(x)$, where $\lambda \in \mathbb{C}$, and $v(x) = v(x_1, \ldots, x_n)$ admits a summable fractional derivative $\left({}^{RL}_{a_1^+}\partial^{\frac{1+\alpha_1}{2}}_{x_1} v \right)(x)$ in the variable x_1, and belongs to $I^{1+\alpha_i}_{a_i^+}(L_1)$ in the variables x_i, with $i = 2, \ldots, n$. The following results generalize Theorem 3.1 and Theorem 3.4 in [10].

Theorem 3.4 *A family of eigenfunctions of the fractional Laplace operator ${}^{RL}\Delta^{\alpha}_{a^+}$ is given by*

$$
\begin{aligned}
u_\lambda(x) = & \; (x_1 - a_1)^{\alpha_1 - 1} \; E_{1+\alpha_1, \alpha_1} \left(-(x_1 - a_1)^{1+\alpha_1} \left({}^{RL}\widehat{\Delta}^{\alpha}_{a^+} - \lambda \right) \right) f_0(\widehat{x}) \\
& + (x_1 - a_1)^{\alpha_1} \; E_{1+\alpha_1, 1+\alpha_1} \left(-(x_1 - a_1)^{1+\alpha_1} \left({}^{RL}\widehat{\Delta}^{\alpha}_{a^+} - \lambda \right) \right) f_1(\widehat{x}),
\end{aligned} \tag{39}
$$

where $f_0(\widehat{x}) = \left(I^{1-\alpha_1}_{a_1^+} u \right)(a_1, \widehat{x})$ and $f_1(\widehat{x}) = \left(\partial^{\alpha_1}_{x_1^+} u \right)(a_1, \widehat{x})$.

The proof of Theorem 3.4 is similar to the proof of Theorem 3.1, however, it takes into account the composition rule (7). For the case $n=3$, see [10].

Corollary 3.5 *A family of fundamental solutions for the fractional Laplace operator* ${}^{RL}\Delta^{\alpha}_{a^+}$ *is obtained by considering* $\lambda = 0$ *in (39):*

$$\begin{aligned}
{}^{RL}G^{\alpha}_{+}(x) &= u_0(x) \\
&= (x_1-a_1)^{\alpha_1-1}\, E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^{RL}\widehat{\Delta}^{\alpha}_{a^+}\right) f_0(\widehat{x}) \\
&\quad + (x_1-a_1)^{\alpha_1}\, E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^{RL}\widehat{\Delta}^{\alpha}_{a^+}\right) f_1(\widehat{x}),
\end{aligned} \tag{40}$$

where $f_0(\widehat{x}) = \left(I^{1-\alpha_1}_{a_1^+} u\right)(a_1,\widehat{x})$ *and* $f_1(\widehat{x}) = \left(\partial^{\alpha_1}_{x_1^+} u\right)(a_1,\widehat{x})$.

For the fractional Dirac operator ${}^{RL}\mathcal{D}^{\alpha}_{a^+}$ we can obtain a family of fundamental solutions by applying the operator ${}^{RL}\mathcal{D}^{\alpha}_{a^+}$ to the family of fundamental solutions of the operator ${}^{RL}\Delta^{\alpha}_{a^+}$.

Theorem 3.6 *A family of fundamental solutions of the fractional Dirac operator* ${}^{RL}\mathcal{D}^{\alpha}_{a^+}$ *(acting on the left or on the right) is given by*

$$ {}^{RL}\mathcal{G}^{\alpha}_{+}(x) = \sum_{i=1}^{n} e_i \left({}^{RL}\mathcal{G}^{\alpha}_{+}\right)_i (x), \tag{41}$$

where

$$\begin{aligned}
\left({}^{RL}\mathcal{G}^{\alpha}_{+}\right)_1 (x) &= (x_1-a_1)^{\frac{\alpha_1-3}{2}} E_{1+\alpha_1,\frac{\alpha_1-1}{2}}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^{RL}\widehat{\Delta}^{\alpha}_{a^+}\right) f_0(\widehat{x}) \\
&\quad + (x_1-a_1)^{\frac{\alpha_1-1}{2}} E_{1+\alpha_1,\frac{1+\alpha_1}{2}}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^{RL}\widehat{\Delta}^{\alpha}_{a^+}\right) f_1(\widehat{x}),
\end{aligned} \tag{42}$$

and for $i=2,\ldots,n$

$$\begin{aligned}
\left({}^{RL}\mathcal{G}^{\alpha}_{+}\right)_i (x) &= (x_1-a_1)^{\alpha_1-1}\left(E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^{RL}\widehat{\Delta}^{\alpha}_{a^+}\right)\ {}^{RL}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}\right) f_0(\widehat{x}) \\
&\quad + (x_1-a_1)^{\alpha_1}\left(E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^{RL}\widehat{\Delta}^{\alpha}_{a^+}\right)\ {}^{RL}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}\right) f_1(\widehat{x})
\end{aligned} \tag{43}$$

with $f_0(\widehat{x}) = \left(I^{1-\alpha_1}_{a_1^+} u\right)(a_1,\widehat{x})$ *and* $f_1(\widehat{x}) = \left(\partial^{\alpha_1}_{x_1^+} u\right)(a_1,\widehat{x})$.

Remark 3.7 *From (40) or (35) it is possible to obtain the fundamental solution of the Euclidean Laplace operator in* $\mathbb{R}^n$ *when* $\alpha = (1,\ldots,1)$. *Let us consider only the Riemann-Liouville case (the Caputo case can be treated similarly). We know that the fundamental solution of the Euclidean Laplace operator in* $\mathbb{R}^n$, $n \geq 3$, *is given (up to a constant) by* $||x-a||^{-(n-2)}$. *The case* $n=2$ *can also be treated but we restrict ourselves to only present the case* $n \geq 3$ *in detail. First we need to obtain the power series expansion of the fundamental solution of the Laplace operator in* $\mathbb{R}^n$. *Considering the binomial series*

$$(1-x)^{-s} = \sum_{p=0}^{+\infty} \begin{pmatrix} s+p-1 \\ p \end{pmatrix} x^p, \qquad |x|<1$$

we obtain

$$\begin{aligned}
\|x-a\|^{-(n-2)} &= \left((x_1-a_1)^2 + \|\widehat{x}-\widehat{a}\|^2\right)^{-\frac{n-2}{2}} \\
&= \|\widehat{x}-\widehat{a}\|^{-(n-2)}\left(1+\frac{(x_1-a_1)^2}{\|\widehat{x}-\widehat{a}\|^2}\right)^{-\frac{n-2}{2}} \\
&= \sum_{p=0}^{+\infty}(-1)^p \begin{pmatrix} \frac{n-2}{2}+p-1 \\ p \end{pmatrix} \frac{(x_1-a_1)^{2p}}{\|\widehat{x}-\widehat{a}\|^{2p+n-2}}
\end{aligned} \tag{44}$$

with $\|\widehat{x}-\widehat{a}\|^2=\sum_{i=2}^n(x_i-a_i)^2$, *and* $\dfrac{(x_1-a_1)^2}{\|\widehat{x}-\widehat{a}\|^2}<1$. *Now, putting* $\alpha=(1,\ldots,1)$ *and* f_1 *the null function in (40) we obtain*

$$u_0(x)=\sum_{p=0}^{+\infty}\frac{(-1)^p\ (x_1-a_1)^{2p}}{\Gamma(2p+1)}\ \widehat{\Delta}^p f_0(\widehat{x}). \tag{45}$$

From a comparison of (44) and (45) we observe that we have to find a function $f_0(\widehat{x})$ *such that*

$$\widehat{\Delta}^p f_0(\widehat{x})\ =\left(\begin{array}{c}\frac{n-2}{2}+p-1\\ p\end{array}\right)\ \frac{(2p)!}{(\|\widehat{x}-\widehat{a}\|)^{2p+n-2}}. \tag{46}$$

We observe that the function $f_0(\widehat{x})=\|\widehat{x}-\widehat{a}\|^{-(n-2)}$ *satisfies (46). First we recall that the p-th powers of the m-dimensional Euclidean Laplace satisfy (see [2, (1.5)])*

$$\Delta^p r^k=\frac{2^{2p}\ \Gamma\left(\frac{k}{2}+1\right)\ \Gamma\left(\frac{k+m}{2}\right)}{\Gamma\left(\frac{k}{2}-p+1\right)\ \Gamma\left(\frac{k+m}{2}-p\right)}\ r^{k-2p}$$

with $r=\|x\|, x\in\mathbb{R}^m$, $p\in\mathbb{N}$ *and* $k\in\mathbb{Z}$. *Therefore, for* $m=n-1$ *and* $k=2-n$ *we obtain*

$$\Delta^p r^{2-n}\ =\frac{2^{2p}\,\Gamma\left(\frac{2-n}{2}+1\right)\ \sqrt{\pi}}{\Gamma\left(\frac{2-n}{2}-p+1\right)\ \Gamma\left(\frac{1}{2}-p\right)}\ r^{2-n-2p}. \tag{47}$$

Comparing (46) and (47) we conclude that we have to show that

$$\left(\begin{array}{c}\frac{n-2}{2}+p-1\\ p\end{array}\right)\ (2p)!\ =\frac{2^{2p}\,\Gamma\left(\frac{2-n}{2}+1\right)\ \sqrt{\pi}}{\Gamma\left(\frac{2-n}{2}-p+1\right)\ \Gamma\left(\frac{1}{2}-p\right)}. \tag{48}$$

This equality is true and can be proved by using well-know relations for the Gamma function and taking into account that

$$\left(\begin{array}{c}\nu\\ k\end{array}\right)=\frac{\Gamma(1+\nu)}{\Gamma(1+k)\,\Gamma(1+\nu-k)}.$$

Therefore, we conclude that the equality (46) is satisfied when $f_0(\widehat{x})=\|\widehat{x}-\widehat{a}\|^{-(n-2)}$. *As expected, on the basis of considering this same function, together with* f_1 *being the null function, we may obtain from (41) and (36) the fundamental solution for the Euclidean Dirac operator in* $\mathbb{R}^n$, $n\geq 3$, *when* $\alpha=(1,\ldots,1)$.

In the following section we introduce fractional versions of the Teodorescu and Cauchy-Bitsadze operators where the kernel of these operators is the fundamental solution ${}^C\mathcal{G}^\alpha_+$. Before we proceed to the development of the operator calculus we present the following auxiliar results.

Theorem 3.8 *For functions* g_0 *and* g_1 *such that*

$$\int_{\widehat{\Omega}}\left|\left({}^C\widehat{\Delta}^\alpha_{a^+}\right)^i g_0(\widehat{x})\right|\,d\widehat{x}<\infty,\qquad \int_{\widehat{\Omega}}\left|\left({}^C\widehat{\Delta}^\alpha_{a^+}\right)^i g_1(\widehat{x})\right|\,d\widehat{x}<\infty,\qquad \forall i\in\mathbb{N}_0,$$

the fundamental solution ${}^C G^\alpha_+$ *belongs to* $L_1(\Omega)$.

Proof: From (35) we have

$$\begin{aligned}\|{}^C G^\alpha_+\|_{L_1(\Omega)} &\leq \left\|E_{1+\alpha_1,1}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^C\widehat{\Delta}^\alpha_{a^+}\right)g_0(\widehat{x})\right\|_{L_1(\Omega)}\\ &\quad+\left\|(x_1-a_1)\ E_{1+\alpha_1,2}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^C\widehat{\Delta}^\alpha_{a^+}\right)g_1(\widehat{x})\right\|_{L_1(\Omega)}\\ &\leq \int_\Omega\left|E_{1+\alpha_1,1}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^C\widehat{\Delta}^\alpha_{a^+}\right)g_0(\widehat{x})\right|\,dx\\ &\quad+\int_\Omega\left|(x_1-a_1)\ E_{1+\alpha_1,2}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^C\widehat{\Delta}^\alpha_{a^+}\right)g_1(\widehat{x})\right|\,dx.\end{aligned}$$

Relying on the series expansion of the Mittag-Leffler function (10), and the fact that the series and integrals that are involved are absolutely convergent, we derive that

$$\begin{aligned}\|{}^C G_+^\alpha\|_{L_1(\Omega)} \leq & \sum_{i=0}^{\infty} \frac{1}{\Gamma((1+\alpha_1)i+1)} \int_{a_1}^{b_1} (x_1-a_1)^{(1+\alpha_1)i}\, dx_1 \int_{\widehat{\Omega}} \left| \left({}^C\widehat{\Delta}_{a^+}^\alpha \right)^i g_0(\widehat{x}) \right| d\widehat{x} \\ & + \sum_{i=0}^{\infty} \frac{1}{\Gamma((1+\alpha_1)i+2)} \int_{a_1}^{b_1} (x_1-a_1)^{(1+\alpha_1)i+1}\, dx_1 \int_{\widehat{\Omega}} \left| \left({}^C\widehat{\Delta}_{a^+}^\alpha \right)^i g_1(\widehat{x}) \right| d\widehat{x},\end{aligned}$$

where $\widehat{\Omega} = \prod_{j=2}^n]a_j, b_j[$, and where g_0, g_1, are chosen such that the integrals over $\widehat{\Omega}$ are finite for each $i = 1, \ldots, n$. Let us denote by C_0 and C_1 the corresponding maximum values over i. Moreover, computing the integrals with respect to x_1 leads to the following inequality

$$\|{}^C G_+^\alpha\|_{L_1(\Omega)} \leq C_0\,(b_1-a_1) \sum_{i=0}^{\infty} \frac{(b_1-a_1)^{(1+\alpha_1)i}}{\Gamma((1+\alpha_1)i+2)} + C_1\,(b_1-a_1)^2 \sum_{i=0}^{\infty} \frac{(b_1-a_1)^{(1+\alpha_1)i}}{\Gamma((1+\alpha_1)i+3)}.$$

Moreover, since $0 < \alpha_1 \leq 1$, we get the final estimate

$$\begin{aligned}\|{}^C G_+^\alpha\|_{L_1(\Omega)} & \leq C_0\,(b_1-a_1) \sum_{i=0}^{\infty} \frac{(b_1-a_1)^{2i}}{\Gamma(i+2)} + C_1\,(b_1-a_1)^2 \sum_{i=0}^{\infty} \frac{(b_1-a_1)^{2i}}{\Gamma(i+3)} \\ & = C_0\, \frac{e^{(b_1-a_1)^2}-1}{b_1-a_1} + C_1\, \frac{e^{(b_1-a_1)^2}-(b_1-a_1)^2-1}{(b_1-a_1)^2}.\end{aligned}$$

The last expression is a finite quantity in view of $a_1 < b_1$.

■

In a very similar way we can prove the following result for the fundamental solution of ${}^C\mathcal{D}_{a^+}^\alpha$.

Theorem 3.9 *For functions g_0 and g_1 such that*

$$\int_{\widehat{\Omega}} \left| \left({}^C\widehat{\Delta}_{a^+}^\alpha \right)^i g_0(\widehat{x}) \right| d\widehat{x} < \infty, \qquad \int_{\widehat{\Omega}} \left| \left({}^C\widehat{\Delta}_{a^+}^\alpha \right)^i g_1(\widehat{x}) \right| d\widehat{x} < \infty, \qquad \forall i \in \mathbb{N}_0,$$

and

$$\int_{\widehat{\Omega}} \left| \left({}^C\widehat{\Delta}_{a^+}^\alpha \right)^i {}^C_{a_i^+}\partial_{x_i}^{\frac{1+\alpha_i}{2}} g_0(\widehat{x}) \right| d\widehat{x} < \infty, \qquad \int_{\widehat{\Omega}} \left| \left({}^C\widehat{\Delta}_{a^+}^\alpha \right)^i {}^C_{a_i^+}\partial_{x_i}^{\frac{1+\alpha_i}{2}} g_1(\widehat{x}) \right| d\widehat{x} < \infty, \qquad \forall i \in \mathbb{N}_0,$$

the fundamental solution ${}^C\mathcal{G}_+^\alpha$ belongs to $L_1(\Omega)$.

4 Fractional Teodorescu and Cauchy-Bitsadze operators

In this section we introduce and study the main properties of the fractional analogues of the classical Teodorescu and Cauchy-Bitsadze operators described in [17]. We start by proving fractional analogues of the Stokes formula and the Borel-Pompeiu formula in a rectangular open rectangular domain of the form $\Omega = \prod_{i=1}^n]a_i, b_i[$. From now on ${}^C\mathcal{D}_{b^-}^\alpha$ denotes the right Caputo fractional Dirac operator, which is given by

$${}^C\mathcal{D}_{b^-}^\alpha = \sum_{i=1}^n e_i\, {}^C_{b_i^-}\partial_{x_i}^{\frac{1+\alpha_i}{2}}, \tag{49}$$

where, for $i = 1, \ldots, n$, the partial derivative ${}^C_{b_i^-}\partial_{x_i}^{\frac{1+\alpha_i}{2}}$ is the right Caputo fractional derivative (6) of order $\frac{1+\alpha_i}{2}$ with respect to the variable $x_i \in]a_i, b_i[$.

Theorem 4.1 *Let $f, g \in AC^1(\Omega) \cap AC(\overline{\Omega})$. Then the following fractional Stokes formula holds*

$$\int_\Omega \left[-\left(f\ {}^C\mathcal{D}_{b^-}^\alpha \right)(x)\, g(x) + f(x)\, \left({}^{RL}\mathcal{D}_{a^+}^\alpha g \right)(x) \right] dx = \int_{\partial\Omega} f(x)\, d\sigma(x)\, \left(I_{a^+}^\alpha g \right)(x), \tag{50}$$

where $d\sigma(x) = n(x)\, d\Omega$, with $n(x)$ being the outward pointing unit normal vector at $x \in \partial\Omega$, where $d\Omega$ is the classical surface element, and dx represents the n-dimensional volume element.

Before we give a proof of theorem we observe that in (50) the operator ${}^{C}\mathcal{D}^{\alpha}_{b^-}$ acts on the right and the operator ${}^{RL}\mathcal{D}^{\alpha}_{a^+}$ acts on the left, which is specific of the Clifford analysis setting because of the lack of commutativity.

Proof: Suppose that f and g satisfy the above mentioned conditions. From (49) and (6) we obtain that

$$\begin{aligned}\int_\Omega -\left(f\ {}^{C}\mathcal{D}^{\alpha}_{b^-}\right)(x)\, g(x)\, dx &= \sum_{i=1}^{n}\int_\Omega \left(f\, I_{b_i^-}^{\frac{1-\alpha_i}{2}}\partial_{x_i}\right)(x)\, e_i\, g(x)\, dx\\ &= \sum_{i=1}^{n}\sum_{A}\int_\Omega \left(f_A\, I_{b_i^-}^{\frac{1-\alpha_i}{2}}\partial_{x_i}\right)(x)\, e_A\, e_i\, g(x)\, dx. \end{aligned} \tag{51}$$

Concerning the integral appearing in (51) we have

$$\int_\Omega \left(f_A\, I_{b_i^-}^{\frac{1-\alpha_i}{2}}\partial_{x_i}\right)(x)\, e_A\, e_i\, g(x)\, dx = \frac{1}{\Gamma\left(\frac{1-\alpha_i}{2}\right)}\int_{\Omega^*}\int_{a_i}^{b_i}\int_{x_i}^{b_i}(w-x_i)^{-\frac{1+\alpha_i}{2}}\left(\partial_{x_i}f_A\right)(x_i^*, w)\, e_A\, e_i\, dw\ g(x)\, dx_i\, dx_i^*, \tag{52}$$

where $\Omega^* = \prod_{k=1,k\neq i}^{n}]a_i, b_i[$, $x_i^* = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and where $\left(\partial_{x_i}f_A\right)(x_i^*, w)$ means that after differentiation, the variable x_i is replaced by w, while the remaining variables remain unchanged. Changing the order of integration in the two inner integrals and relying on (1), we obtain that the right-hand side of (52) equals to

$$\begin{aligned}&\frac{1}{\Gamma\left(\frac{1-\alpha_i}{2}\right)}\int_{\Omega^*}\int_{a_i}^{b_i}\left(\partial_{x_i}f_A\right)(x_i^*, w)\, e_A\, e_i\int_{a_i}^{w}(w-x_i)^{-\frac{1+\alpha_i}{2}}\ g(x)\, dx_i\ dw\, dx_i^*\\ &\qquad = \int_\Omega \left(\partial_{x_i}f_A\right)(x)\, e_A\, e_i\left(I_{a_i^+}^{\frac{1-\alpha_i}{2}}g\right)(x)\, dx. \end{aligned} \tag{53}$$

Hence, inserting (53) into (51) we conclude that

$$\int_\Omega -\left(f\ {}^{C}\mathcal{D}^{\alpha}_{b^-}\right)(x)\, g(x)\, dx \ = \ \sum_{i=1}^{n}\sum_{A}\int_\Omega \left(\partial_{x_i}f_A\right)(x)\, e_A\, e_i\left(I_{a_i^+}^{\frac{1-\alpha_i}{2}}g\right)(x)\, dx. \tag{54}$$

Applying now the classical Stokes formula (see [8]) to the right-hand side of (54) and applying (3), we get

$$\begin{aligned}\int_\Omega -\left(f\ {}^{C}\mathcal{D}^{\alpha}_{b^-}\right)(x)\, g(x)\, dx &= \sum_{i=1}^{n}\sum_{A}\left[\int_{\partial\Omega} f_A(x)\, e_A\, d\sigma(x)\left(I_{a_i^+}^{\frac{1-\alpha_i}{2}}g\right)(x) - \int_\Omega f_A(x)\, e_A\, e_i\left(\partial_{x_i}I_{a_i^+}^{\frac{1-\alpha_i}{2}}g\right)(x)\, dx\right]\\ &= \sum_{i=1}^{n}\int_{\partial\Omega} f(x)\, d\sigma(x)\left(I_{a_i^+}^{\frac{1-\alpha_i}{2}}g\right)(x)\ -\ \sum_{i=1}^{n}\int_\Omega f(x)\, e_i\left({}^{RL}_{a_i^+}\partial_{x_i}^{\frac{1+\alpha_i}{2}}g\right)(x)\, dx. \end{aligned}$$

Therefore, from (49) we obtain the following fractional Stokes formula

$$\int_\Omega \left[-\left(f\ {}^{C}\mathcal{D}^{\alpha}_{b^-}\right)(x)\, g(x) + f(x)\ \left({}^{RL}\mathcal{D}^{\alpha}_{a^+}g\right)(x)\right]\ dx \ = \ \int_{\partial\Omega} f(x)\, d\sigma(x)\ \left(I^{\alpha}_{a^+}g\right)(x),$$

where $\left(I^{\alpha}_{a^+}g\right)(x) = \displaystyle\sum_{i=1}^{n}\left(I_{a_i^+}^{\frac{1-\alpha_i}{2}}g\right)(x)$.

■

We notice that the Stokes's formula in the classical Clifford analysis setting has the form

$$\int_\Omega \left[(f\,\mathcal{D})(x)\, g(x) + f(x)\ (\mathcal{D}\, g)(x)\right]\ dx = \int_{\partial\Omega} f(x)\, d\sigma(x)\, g(x)\,, \tag{55}$$

where $\mathcal{D}$ is the Euclidean Dirac operator. However, in the fractional Clifford analysis setting we obtain a more complicatedly kind of "double duality" relation. On the one hand the formula involves both the Caputo and Riemann-Liouville derivatives, and on the other hand it also involves left and right derivatives. It is also possible to obtain other versions of the fractional Stokes's formula. For example, if we consider in (51) the operator ${}^{RL}\mathcal{D}^{\alpha}_{a^+}$ then we obtain the following alternative version of the Stokes's formula:

$$\int_\Omega \left[\left(f\, {}^{RL}\mathcal{D}^{\alpha}_{a^+}\right)(x)\, g(x) - f(x)\ \left({}^{C}\mathcal{D}^{\alpha}_{b^-}g\right)(x)\right]\ dx = \int_{\partial\Omega}\left(I^{\alpha}_{a^+}f\right)(x)\, d\sigma(x)\, g(x).$$

Before we deduce our fractional Borel-Pompeiu formula, we need to understand the behaviour of the fractional Dirac operator ${}^C\mathcal{D}^{\alpha}_{b^-}$ when the argument of the function f in (50) is translated and reflected. Denoting the translation operator by $\mathcal{T}_\theta\, f(y) := f(\theta + y)$ and the reflection operator by $\mathcal{R}_y\, f(y) := f(-y)$, and applying the definitions of the right and left Caputo fractional derivatives presented in (6) and (5) we can deduce the following relation (where the derivative is with respect to the variable y):

$$(f(\theta-y))\, {}^C\mathcal{D}^{\alpha}_{(\theta-a)^-} \;= (\mathcal{T}_\theta\, \mathcal{R}_y\, f(y))\, {}^C\mathcal{D}^{\alpha}_{(\theta-a)^-} \;= -\mathcal{T}_\theta\, \mathcal{R}_y\, \left(f(y)\, {}^C\mathcal{D}^{\alpha}_{a^+}\right) \;= -\left(f\, {}^C\mathcal{D}^{\alpha}_{a^+}\right)(\theta-y). \tag{56}$$

Replacing f by ${}^C\mathcal{G}^{\alpha}_{+}(x+a-y)$ in (50) and integrating with respect to the variable y, we obtain the following fractional Borel-Pompeiu formula and fractional Cauchy's integral formula.

Corollary 4.2 *Let $g \in AC^1(\Omega) \cap AC(\overline{\Omega})$. Then the following fractional Borel-Pompeiu formula holds*

$$-\int_{\Omega} {}^C\mathcal{G}^{\alpha}_{+}(x+a-y)\, \left({}^{RL}\mathcal{D}^{\alpha}_{a^+} g\right)(y)\, dy + \int_{\partial\Omega} {}^C\mathcal{G}^{\alpha}_{+}(x+a-y)\, d\sigma(y)\, \left(I^{\alpha}_{a^+} g\right)(y) = g(x). \tag{57}$$

Moreover, if $g \in \ker\left({}^{RL}\mathcal{D}^{\alpha}_{a^+}\right)$, then we obtain the fractional Cauchy's integral formula

$$\int_{\partial\Omega} {}^C\mathcal{G}^{\alpha}_{+}(x+a-y)\; d\sigma(y)\; \left(I^{\alpha}_{a^+} g\right)(y) = g(x). \tag{58}$$

Proof: Note that ${}^C\mathcal{G}^{\alpha}_{+}(y)$ is the fundamental solution of ${}^C\mathcal{D}^{\alpha}_{a^+}$ defined only for $y_i > a_i$, $i = 1, \ldots, n$ and satisfies $\left({}^C\mathcal{D}^{\alpha}_{a^+}\, {}^C\mathcal{G}^{\alpha}_{+}\right)(y) = \delta(y-a)$. First we replace f by ${}^C\mathcal{G}^{\alpha}_{+}(x+a-y) = \mathcal{T}_{x+a}\, \mathcal{R}_y\, {}^C\mathcal{G}^{\alpha}_{+}(y)$ in (50). Since ${}^C\mathcal{G}^{\alpha}_{+}(x+a-y)$ is defined only for $y_i < x_i$, $i = 1, \ldots, n$, due to translations and reflections, the operator ${}^C\mathcal{D}^{\alpha}_{b^-}$ in (50) is replaced by ${}^C\mathcal{D}^{\alpha}_{x^-}$. Thus, we have

$$-\int_{\Omega} {}^C\mathcal{G}^{\alpha}_{+}(x+a-y)\; {}^C\mathcal{D}^{\alpha}_{x^-}\, g(y)\, dy + \int_{\Omega} {}^C\mathcal{G}^{\alpha}_{+}(x+a-y)\; \left({}^{RL}\mathcal{D}^{\alpha}_{a^+} g\right)(y)\, dy = \int_{\partial\Omega} {}^C\mathcal{G}^{\alpha}_{+}(x+a-y)\, d\sigma(y)\; \left(I^{\alpha}_{a^+} g\right)(y).$$

Applying (56) with $\theta = x + a$, leads to

$$\int_{\Omega} \delta(x-y)\, g(y)\, dy + \int_{\Omega} {}^C\mathcal{G}^{\alpha}_{+}(x+a-y)\; \left({}^{RL}\mathcal{D}^{\alpha}_{a^+} g\right)(y)\, dy = \int_{\partial\Omega} {}^C\mathcal{G}^{\alpha}_{+}(x+a-y)\, d\sigma(y)\; \left(I^{\alpha}_{a^+} g\right)(y).$$

The previous expression leads to the fractional Borel-Pompeiu formula

$$g(x) = -\int_{\Omega} {}^C\mathcal{G}^{\alpha}_{+}(x+a-y)\; \left({}^{RL}\mathcal{D}^{\alpha}_{a^+} g\right)(y)\, dy + \int_{\partial\Omega} {}^C\mathcal{G}^{\alpha}_{+}(x+a-y)\, d\sigma(y)\; \left(I^{\alpha}_{a^+} g\right)(y).$$

Additionally, if $g \in \ker\left({}^{RL}\mathcal{D}^{\alpha}_{a^+}\right)$ then the first integral of the right-hand side of the preceding expression is equal to zero. Therefore, we arrive at the fractional version of Cauchy's integral formula stated in (58).

■

As a consequence of the "double duality" mentioned previously it is possible to deduce alternative versions of the previous Borel-Pompeiu formula.

Remark 4.3 *In the case $\alpha = (1, \ldots, 1)$, $a = (0, \ldots, 0)$, $g_1 \equiv 0$, $g_0(\widehat{x}) = \|\widehat{x}\|^{-(n-2)}$ in (50), (57), and (58), and taking into account that $\left({}^C\mathcal{D}^{1}_{b^-} g\right)(x) = -\left(\mathcal{D}g\right)(x)$, $\left({}^{RL}\mathcal{D}^{1}_{a^+} g\right)(x) = \left(\mathcal{D}g\right)(x)$, and $\left(I^{0}_{b^-} g\right)(x) = g(x)$ we obtain the classical Stokes formula (55), the Borel-Pompeiu formula, and Cauchy's integral formula presented in [8, Sect. 2.1] and [17, Sect. 3.2].*

From (57) we may introduce the following definition.

Definition 4.4 *Let $g \in AC^1(\Omega)$. Then the linear integral operators*

$$\left({}^C T^{\alpha} g\right)(x) = -\int_{\Omega} {}^C\mathcal{G}^{\alpha}_{+}(x+a-y)\, g(y)\; dy \tag{59}$$

and

$$\left({}^C F^{\alpha} g\right)(x) = \int_{\partial\Omega} {}^C\mathcal{G}^{\alpha}_{+}(x+a-y)\, d\sigma(y)\, \left(I^{\alpha}_{a^+} g\right)(y) \tag{60}$$

are called the fractional Teodorescu and Cauchy-Bitsadze operator, respectively.

Remark 4.5 *In the case $\alpha = (1, \ldots, 1)$, $a = (0, \ldots, 0)$, $g_1 \equiv 0$, and $g_0(\widehat{x}) = \|\widehat{x}\|^{-(n-2)}$, the operators ${}^{C}T^{\alpha}$ and ${}^{C}F^{\alpha}$ coincide with the usual and well-known classical operators defined in [17, Def. 3.1, 3.25].*

The previous definition allows us to rewrite (57) in the alternative form

$$\left({}^{C}T^{\alpha}\ {}^{RL}\mathcal{D}^{\alpha}_{a^+} g\right)(x) + \left({}^{C}F^{\alpha} g\right)(x) = g(x), \qquad x \in \Omega.$$

Now we study some properties of the fractional integral operators ${}^{C}T^{\alpha}$ and ${}^{C}F^{\alpha}$. We point out that in all the forthcoming results the parameter p referring to the L_p-space belongs to the interval $\left]1, \frac{2}{1-\alpha^*}\right[$, with $\alpha^* = \min_{1\le i\le n}\{\alpha_i\}$. This specific range of p results from the application of Theorem 2.6 to the fractional integrals of order $\frac{1-\alpha_i}{2}$, with $i = 1, \ldots, n$, that appear in the definition of the fractional differential operators. Moreover, the parameter q of the L_q-space must be chosen such that $q = \frac{2p}{2-(1-\alpha^*)p}$, according to Theorem 2.6. If $\alpha = (1, \ldots, 1)$, then we conclude that $p \in]1, \infty[$ and $q = p$, as it occurs in the classical setting (see [17, Ch.3]). Before we deduce two properties of the fractional integral operators (59) and (60), we need to understand the behaviour of our fractional derivatives when the argument of the function over which we apply the derivatives is only translated. Denoting the translation by $\mathcal{T}_\theta\, f(x) := f(x+\theta)$, and using the definition of the left Caputo fractional derivative presented in (5), we can deduce the following relation (where the derivative is with respect to the variable x):

$$\left({}^{C}\mathcal{D}^{\alpha}_{(-\theta+a)^+}\, f(x+\theta)\right) = \left({}^{C}\mathcal{D}^{\alpha}_{(-\theta+a)^+}\, \mathcal{T}_\theta\, f(x)\right) = \mathcal{T}_\theta\, \left({}^{C}\mathcal{D}^{\alpha}_{a^+}\, f(x)\right) = \left({}^{C}\mathcal{D}^{\alpha}_{a^+}\, f\right)(x+\theta). \tag{61}$$

Theorem 4.6 *The fractional operator ${}^{C}T^{\alpha}$ is the right inverse of ${}^{C}\mathcal{D}^{\alpha}_{a^+}$, i.e., for $g \in L_p(\Omega)$, with $p \in \left]1, \frac{2}{1-\alpha^*}\right[$ and*
$\alpha^* = \min_{1\le i\le n}\{\alpha_i\}$, we have

$$\left({}^{C}\mathcal{D}^{\alpha}_{a^+}\ {}^{C}T^{\alpha} g\right)(x) = g(x).$$

Proof: Note that ${}^{C}\mathcal{G}^{\alpha}_{+}(x)$ is the fundamental solution of ${}^{C}\mathcal{D}^{\alpha}_{a^+}$ defined only for $x_i > a_i$, $i = 1, \ldots, n$ and satisfies $\left({}^{C}\mathcal{D}^{\alpha}_{a^+}\, {}^{C}\mathcal{G}^{\alpha}_{+}\right)(x) = \delta(x-a)$. With respect to the variable x, the function ${}^{C}\mathcal{G}^{\alpha}_{+}(x+a-y)$ is defined only for $x_i > y_i$, $i = 1, \ldots, n$, therefore, the operator ${}^{C}\mathcal{D}^{\alpha}_{a^+}$ is replaced by ${}^{C}\mathcal{D}^{\alpha}_{y^+}$. Taking into account the definition of ${}^{C}T^{\alpha}$ given in (59) and the relation (61) with $\theta = a - y$, we obtain

$$\begin{aligned}
\left({}^{C}\mathcal{D}^{\alpha}_{a^+}\ {}^{C}T^{\alpha} g\right)(x) &= -\int_\Omega {}^{C}\mathcal{D}^{\alpha}_{y^+}\ {}^{C}\mathcal{G}^{\alpha}_{+}(x+a-y)\, g(y)\, dy \\
&= -\int_\Omega {}^{C}\mathcal{D}^{\alpha}_{y^+}\left(\mathcal{T}_{a-y}\, {}^{C}\mathcal{G}^{\alpha}_{+}(x)\right) g(y)\, dy \\
&= -\int_\Omega \mathcal{T}_{a-y}\left({}^{C}\mathcal{D}^{\alpha}_{a^+}\, {}^{C}\mathcal{G}^{\alpha}_{+}(x)\right) g(y)\, dy \\
&= -\int_\Omega \mathcal{T}_{a-y}\, \delta(x-a)\, g(y)\, dy \\
&= -\int_\Omega \delta(x-y)\, g(y)\, dy \\
&= g(x).
\end{aligned}$$

■

In a similar way as in [18], we introduce the fractional Sobolev space $W^{\alpha,p}_{a^+}(\Omega)$, specifically adapted to our problem, with the norm $\|\cdot\|_{W^{\alpha,p}_{a^+}(\Omega)}$ given by

$$\|f\|^p_{W^{\alpha,p}_{a^+}(\Omega)} := \|f\|^p_{L_p(\Omega)} + \sum_{k=1}^{n} \left\| {}^{C}_{a_k^+}\partial^{\frac{1+\alpha_k}{2}}_{x_k}\, f \right\|^p_{L_p(\Omega)},$$

where $\|\cdot\|_{L_p(\Omega)}$ is the usual L_p-norm in Ω, and $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_k \in]0, 1]$, $k = 1, \ldots, n$.

Theorem 4.7 *The fractional operator ${}^C F^\alpha$ maps $W_{a^+}^{\alpha-\frac{1}{p},p}(\partial\Omega)$−functions to functions belonging to the kernel of ${}^C\mathcal{D}_{a^+}^\alpha$, i.e., the fractional operator ${}^C F^\alpha$ satisfies $\left({}^C\mathcal{D}_{a^+}^\alpha\, {}^C F^\alpha g\right)(x) = 0$, for every $g \in W_{a^+}^{\alpha-\frac{1}{p},p}(\partial\Omega)$, with $p \in \left]1, \frac{2}{1-\alpha^*}\right[$ and $\alpha^* = \min_{1\leq i\leq n}\{\alpha_i\}$.*

Proof: By the same reasonings used Theorem 4.6, we have from (60) that

$$\begin{aligned}
\left({}^C\mathcal{D}_{a^+}^\alpha\, {}^C F^\alpha g\right)(x) &= \int_{\partial\Omega} {}^C\mathcal{D}_{y^+}^\alpha\, {}^C\mathcal{G}_+^\alpha(x+a-y)\, d\sigma(y)\, \left(I_{a^+}^\alpha g\right)(y) \\
&= \int_{\partial\Omega} {}^C\mathcal{D}_{y^+}^\alpha \left(\mathcal{T}_{a-y}\, {}^C\mathcal{G}_+^\alpha(x)\right) d\sigma(y)\, \left(I_{a^+}^\alpha g\right)(y) \\
&= \int_{\partial\Omega} \mathcal{T}_{a-y} \left({}^C\mathcal{D}_{a^+}^\alpha\, {}^C\mathcal{G}_+^\alpha(x)\right) d\sigma(y)\, \left(I_{a^+}^\alpha g\right)(y) \\
&= \int_{\partial\Omega} \mathcal{T}_{a-y}\, \delta(x-a)\, d\sigma(y)\, \left(I_{a^+}^\alpha g\right)(y) \\
&= \int_{\partial\Omega} \delta(x-y)\, d\sigma(y)\, \left(I_{a^+}^\alpha g\right)(y) \\
&= 0.
\end{aligned}$$

Note that the validity of the last equality is due to the fact that $x \in \Omega$ and $y \in \partial\Omega$, i.e., the difference $x-y$ is always non-zero.

■

Now we present some mapping properties of the fractional operators ${}^C T^\alpha$ and ${}^C F^\alpha$.

Theorem 4.8 *The operator ${}^C T^\alpha$ is bounded from $L_p(\Omega)$ to $L_p(\Omega)$, with $p \in \left]1, \frac{2}{1-\alpha^*}\right[$ and $\alpha^* = \min_{1\leq i\leq n}\{\alpha_i\}$.*

Proof: Under the previous conditions, and in view of the Young's inequality for convolutions (see Theorem 1.4 in [26]) and Theorem 3.9, we obtain

$$\left\|{}^C T^\alpha g\right\|_{L_p(\Omega)} = \left\|{}^C\mathcal{G}_+^\alpha * g\right\|_{L_p(\Omega)} \leq \left\|{}^C\mathcal{G}_+^\alpha\right\|_{L_1(\Omega)} \|g\|_{L_p(\Omega)},$$

which leads to our result.

■

Remark 4.9 *Considering $\alpha = (1,\ldots,1)$, $a = (0,\ldots,0)$, $g_1 \equiv 0$, and $g_0(\widehat{x}) = \|\widehat{x}\|^{-(n-2)}$ we obtain Theorem 3.9 in [17].*

Now we want to study the derivatives of ${}^C T^\alpha$. Before we do that we present an auxiliar result where we calculate the partial fractional derivatives of ${}^C\mathcal{G}_+^\alpha$.

Theorem 4.10 *The partial fractional derivatives of the fundamental solution ${}^C\mathcal{D}_{a^+}^\alpha$ are given by*

$$\begin{aligned}
\left({}^C_{a_1^+}\partial_{x_1}^{\frac{1+\alpha_1}{2}}\, {}^C\mathcal{G}_+^\alpha\right)(x) = e_1 \Big[& (x_1-a_1)^{-(1+\alpha_1)}\, E_{1+\alpha_1,-\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^C\widehat{\Delta}_{a^+}^\alpha\right) g_0(\widehat{x}) \\
& +(x_1-a_1)^{-\alpha_1}\, E_{1+\alpha_1,1-\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^C\widehat{\Delta}_{a^+}^\alpha\right) g_1(\widehat{x}) \Big] \\
& + \sum_{i=2}^n e_i \left[(x_1-a_1)^{-\frac{1+\alpha_1}{2}}\, E_{1+\alpha_1,\frac{1-\alpha_1}{2}}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^C\widehat{\Delta}_{a^+}^\alpha\right) {}^C_{a_i^+}\partial_{x_i}^{\frac{1+\alpha_i}{2}}\, g_0(\widehat{x}) \right. \\
& \left. +(x_1-a_1)^{\frac{1-\alpha_1}{2}}\, E_{1+\alpha_1,\frac{3-\alpha_1}{2}}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^C\widehat{\Delta}_{a^+}^\alpha\right) {}^C_{a_i^+}\partial_{x_i}^{\frac{1+\alpha_i}{2}}\, g_1(\widehat{x}) \right],
\end{aligned} \tag{62}$$

and for $k = 2, \dots, n$

$$\left({}^C_{a_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}} \, {}^C\mathcal{G}_+^\alpha \right)(x) = e_1 \left[(x_1 - a_1)^{-\frac{1+\alpha_1}{2}} E_{1+\alpha_1, \frac{1-\alpha_1}{2}} \left(-(x_1 - a_1)^{1+\alpha_1} \; {}^C\widehat{\Delta}_{a^+}^\alpha \right) {}^C_{a_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}} g_0(\widehat{x}) \right.$$

$$\left. + (x_1 - a_1)^{\frac{1-\alpha_1}{2}} E_{1+\alpha_1, \frac{3-\alpha_1}{2}} \left(-(x_1 - a_1)^{1+\alpha_1} \; {}^C\widehat{\Delta}_{a^+}^\alpha \right) {}^C_{a_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}} g_1(\widehat{x}) \right]$$

$$+ \sum_{i=2}^n e_i \left[E_{1+\alpha_1, 1} \left(-(x_1 - a_1)^{1+\alpha_1} \; {}^C\widehat{\Delta}_{a^+}^\alpha \right) {}^C_{a_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}} \, {}^C_{a_i^+}\partial_{x_i}^{\frac{1+\alpha_i}{2}} g_0(\widehat{x}) \right.$$

$$\left. + (x_1 - a_1) E_{1+\alpha_1, 2} \left(-(x_1 - a_1)^{1+\alpha_1} \; {}^C\widehat{\Delta}_{a^+}^\alpha \right) {}^C_{a_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}} \, {}^C_{a_i^+}\partial_{x_i}^{\frac{1+\alpha_i}{2}} g_1(\widehat{x}) \right]. \tag{63}$$

Proof: Let us start with the proof of (62). Applying the operator ${}^C_{a_1^+}\partial_{x_1}^{\frac{1+\alpha_1}{2}}$ to ${}^C\mathcal{G}_+^\alpha$ and relying on (13) we obtain the expression (62). Concerning (63) the deduction is even more direct because the operator ${}^C_{a_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}}$ only acts on the functions g_0 and g_1.

■

Let us now study the derivatives of ${}^C T^\alpha$.

Theorem 4.11 *Let* $g \in L_q(\Omega)$*, with* $q = \frac{2p}{2-(1-\alpha^*)p}$*,* $p \in \left]1, \frac{2}{1-\alpha^*}\right[$*, and* $\alpha^* = \min_{1 \le i \le n}\{\alpha_i\}$*. The fractional partial derivatives of* ${}^C T^\alpha$ *with respect to* x_k *satisfy the mapping property*

$${}^C_{a_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}} \left({}^C T^\alpha g \right) : \; L_q(\Omega) \longrightarrow L_q(\Omega), \qquad k = 1, 2, \dots, n.$$

Proof: Since for $k = 1, \dots, n$ we have

$$\begin{aligned} \left({}^C_{a_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}} \, {}^C T^\alpha g \right)(x) &= -\int_\Omega {}^C_{y_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}} \, {}^C\mathcal{G}_+^\alpha(x + a - y) \, g(y) \, dy \\ &= -\int_\Omega {}^C_{y_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}} \left(\mathcal{T}_{a-y} \, {}^C\mathcal{G}_+^\alpha(x) \right) g(y) \, dy \\ &= -\int_\Omega \mathcal{T}_{a-y} \left({}^C_{a_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}} \, {}^C\mathcal{G}_+^\alpha(x) \right) g(y) \, dy, \end{aligned} \tag{64}$$

then to study the derivatives of the operator ${}^C T^\alpha$ it suffices to study the convolution terms (64) (see [24]). The expression for the kernel of this convolution corresponds to the expressions (62) and (63). These kernels can be proved to be L_1-functions in a similar way as it was done in the proof of Theorem 3.8, with g_0 and g_1 like in Theorem 3.8. This fact, combined with Young's inequality for convolutions (see Theorem 1.4 in [26]), leads to

$$\left\| {}^C_{a_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}} \left({}^C T^\alpha g \right) \right\|_{L_q(\Omega)} = \left\| \left({}^C_{a_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}} \, {}^C\mathcal{G}_+^\alpha \right) * g \right\|_{L_q(\Omega)} \le \left\| {}^C_{a_k^+}\partial_{x_k}^{\frac{1+\alpha_k}{2}} \, {}^C\mathcal{G}_+^\alpha \right\|_{L_1(\Omega)} \|g\|_{L_q(\Omega)}$$

which in turn implies our result.

■

Remark 4.12 *Considering* $\alpha = (1, \dots, 1)$*,* $a = (0, \dots, 0)$*,* $g_1 \equiv 0$*, and* $g_0(\widehat{x}) = \|\widehat{x}\|^{-(n-2)}$ *the previous theorem reduces to Theorem 3.8 in [17].*

Theorems 4.8 and 4.11 allow us to prove the continuity of ${}^C T^\alpha$.

Theorem 4.13 *Let* $q = \frac{2p}{2-(1-\alpha^*)p}$*,* $p \in \left]1, \frac{2}{1-\alpha^*}\right[$*, and* $\alpha^* = \min_{1 \le i \le n}\{\alpha_i\}$*. The operator* ${}^C T^\alpha : L_q(\Omega) \to W_{a^+}^{\alpha,q}(\Omega)$ *is continuous.*

This result can be obtained as direct consequence of Theorem 4.8 and Theorem 4.11 and, therefore, we omit a detailed proof. Now we study the mapping properties of ${}^C F^\alpha$.

Theorem 4.14 *Let $q = \frac{2p}{2-(1-\alpha^*)p}$, $p \in \left]1, \frac{2}{1-\alpha^*}\right[$, and $\alpha^* = \min_{1\leq i\leq n}\{\alpha_i\}$. The operator ${}^{C}F^{\alpha}$ acts continuously on $W_{a^+}^{\alpha-\frac{1}{p},p}(\partial\Omega)$, more precisely, the operator*

$$ {}^{C}F^{\alpha} : W_{a^+}^{\alpha-\frac{1}{p},p}(\partial\Omega) \to W_{a^+}^{\alpha,q}(\Omega) \cap \ker\left({}^{C}\mathcal{D}_{a^+}^{\alpha}\right) $$

is continuous.

Proof: For a function $f \in W_{a^+}^{\alpha-\frac{1}{p},p}(\partial\Omega)$ we can find a function $g \in W_{a^+}^{\alpha,q}(\Omega)$ such that $g = {}^{C}F^{\alpha} f$. Next, by the Borel-Pompeiu formula (57) we may infer that ${}^{C}F^{\alpha} f = \left(I - {}^{C}T^{\alpha}\, {}^{RL}\mathcal{D}_{a^+}^{\alpha}\right) g$. In view of the continuity of ${}^{C}T^{\alpha}$ and the fact that for a function $g \in W_{a^+}^{\alpha,q}(\Omega)$ we have ${}^{RL}\mathcal{D}_{a^+}^{\alpha} g \in W_{a^+}^{\alpha,q}(\Omega)$, and hence we conclude that $(I - {}^{C}T^{\alpha\, RL}\mathcal{D}_{a^+}^{\alpha})g \in W_{a^+}^{\alpha,q}(\Omega)$. By Theorem 4.6 and Theorem 4.7 we have $0 = {}^{C}\mathcal{D}_{a^+}^{\alpha}\, {}^{C}F^{\alpha} f = \left({}^{C}\mathcal{D}_{a^+}^{\alpha}(I - {}^{C}T^{\alpha\, RL}\mathcal{D}_{a^+}^{\alpha})\right) g$. This in turn implies that $g = {}^{C}F^{\alpha} f \in W_{a^+}^{\alpha,q}(\Omega) \cap \ker({}^{C}\mathcal{D}_{a^+}^{\alpha})$ for a function $g \in W_{a^+}^{\alpha,q}(\Omega)$.

■

Remark 4.15 *Considering $\alpha = (1,\ldots,1)$, $a = (0,\ldots,0)$, $g_1 \equiv 0$, and $g_0(\widehat{x}) = \|\widehat{x}\|^{-(n-2)}$ we have that $\ker\left({}^{C}\mathcal{D}_{a^+}^{\alpha}\right)$ reduces to $\ker(\mathcal{D})$ as it happens in [17].*

5 Hodge-type decomposition

The aim of this section is to obtain a Hodge-type decomposition and to present an immediate application of this decomposition for the resolution of boundary value problems involving the fractional Laplace operator. To realize this we need first the following lemma.

Lemma 5.1 *Let u be in $L_1(\Omega)$, and v has a summable fractional derivative $\left({}^{C}_{a_1^+}\partial_{x_1}^{1+\alpha_1} v\right)(x)$ in the variable x_1, and belongs to $I_{a_i^+}^{1+\alpha_i}(L_1)$ in the variables x_i, with $i = 2,\ldots,n$. The solution $v(x)$ of the Poisson equation*

$$ {}^{C}\Delta_{a^+}^{\alpha} v(x) = u(x) \tag{65} $$

is given in the operator form by

$$ \begin{aligned} v(x) &= E_{1+\alpha_1,1}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^{C}\widehat{\Delta}_{a^+}^{\alpha}\right) g_0(\widehat{x}) + (x_1-a_1)\, E_{1+\alpha_1,2}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^{C}\widehat{\Delta}_{a^+}^{\alpha}\right) g_1(\widehat{x}) \\ &\quad + \sum_{m=0}^{\infty}\left(-\ {}^{C}\widehat{\Delta}_{a^+}^{\alpha}\right)^m \left(I_{a_1^+}^{(1+\alpha_1)(m+1)} u\right)(x), \end{aligned} \tag{66} $$

where the functions g_0 and g_1 are the Cauchy initial conditions given by

$$ g_0(\widehat{x}) = v(a_1,\widehat{x}) \qquad \text{and} \qquad g_1(\widehat{x}) = v'_{x_1}(a_1,\widehat{x}). \tag{67} $$

Proof: The proof follows the same reasoning of the deduction of (33). Applying successively the fractional integrals $I_{a_j^+}^{1+\alpha_j}$, with $j = 1,\ldots,n$, to both sides of (65), applying Fubini's theorem, and rearranging the terms, we get

$$ \begin{aligned} &\left(I_{a_1^+}^{1+\alpha_1} \sum_{k=2}^{n}\prod_{\substack{j=2\\ j\neq k}}^{n} I_{a_j^+}^{1+\alpha_j} v\right)(x) + \left(\prod_{j=2}^{n} I_{a_j^+}^{1+\alpha_j} v\right)(x) \\ &\qquad = \left(\prod_{j=2}^{n} I_{a_j^+}^{1+\alpha_j} g_0\right)(\widehat{x}) + (x_1-a_1)\left(\prod_{j=2}^{n} I_{a_j^+}^{1+\alpha_j} g_1\right)(\widehat{x}) + \left(\prod_{j=1}^{n} I_{a_j^+}^{1+\alpha_j} u\right)(x), \end{aligned} \tag{68} $$

where g_0 and g_1 are the Cauchy initial conditions given in (67). Applying the $(n-1)$-dimensional Laplace transform with respect to $\widehat{x} = (x_2,\ldots,x_n)$ to (68), taking into account the relations (24)-(28), and multiplying by $\prod_{p=2}^{n} s_p^{1+\alpha_p}$ we obtain the following second kind homogeneous integral equation of Volterra type:

$$ \mathcal{V}(x_1,\widehat{s}) + \frac{\sum_{p=2}^{n} s_p^{1+\alpha_p}}{\Gamma(1+\alpha_1)}\int_{a_1}^{x_1}(x_1-t)^{\alpha_1}\,\mathcal{V}(t,\widehat{s})\,dt \ = G(x_1,\widehat{s}) + \left(I_{a_1^+}^{1+\alpha_1}\mathcal{U}\right)(x_1,\widehat{s}), \tag{69} $$

where $G(x_1,\widehat{s}) = \mathfrak{G}_0(\widehat{s}) + (x_1 - a_1)\,\mathfrak{G}_1(\widehat{s})$ and $\mathfrak{G}_k(\widehat{s}) = \mathfrak{L}\{g_k\}(s)$ with $k = 0, 1$. Using (16), we have that the unique solution of (69) in the class of summable functions is:

$$\mathcal{V}(x_1,\widehat{s}) = G(x_1,\widehat{s}) - \frac{\sum_{p=2}^{n} s_p^{1+\alpha_p}}{\Gamma(1+\alpha_1)} \int_{a_1}^{x_1} (x_1-t)^{\alpha_1} E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-t)^{1+\alpha_1} \sum_{p=2}^{n} s_p^{1+\alpha_p}\right) G(t,\widehat{s})\,dt$$

$$+ \left(I_{a_1^+}^{1+\alpha_1}\mathcal{U}\right)(x_1,\widehat{s}) - \frac{\sum_{p=2}^{n} s_p^{1+\alpha_p}}{\Gamma(1+\alpha_1)} \int_{a_1}^{x_1} (x_1-t)^{\alpha_1} E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-t)^{1+\alpha_1} \sum_{p=2}^{n} s_p^{1+\alpha_p}\right) \left(I_{a_1^+}^{1+\alpha_1}\mathcal{U}\right)(x_1,\widehat{s})\,dt. \tag{70}$$

The first two terms in (70) coincide with (30) and are equal to (31) with $\lambda = 0$. Concerning the last two terms in (70), due the convergence of the integrals and the series, we can interchange them and rewrite them in the following way (in the calculations we make a change of the order of integration):

$$\begin{aligned}
&\left(I_{a_1^+}^{1+\alpha_1}\mathcal{U}\right)(x_1,\widehat{s}) + \sum_{k=0}^{+\infty} \frac{\left(-\sum_{p=2}^{n} s_p^{1+\alpha_p}\right)^{k+1}}{\Gamma((1+\alpha_1)k+1+\alpha_1)} \int_{a_1}^{x_1} \mathcal{U}(w,\widehat{s}) \int_{w}^{x_1} (x_1-t)^{(1+\alpha_1)k+\alpha_1} (t-w)^{\alpha_1}\,dt\,dw \\
&= \left(I_{a_1^+}^{1+\alpha_1}\mathcal{U}\right)(x_1,\widehat{s}) + \sum_{k=0}^{+\infty} \frac{\left(-\sum_{p=2}^{n} s_p^{1+\alpha_p}\right)^{k+1}}{\Gamma((1+\alpha_1)k+2(1+\alpha_1))} \int_{a_1}^{x_1} (x-w)^{(1+\alpha_1)k+2\alpha_1+1}\,\mathcal{U}(w,\widehat{s})\,dw \\
&= \left(I_{a_1^+}^{1+\alpha_1}\mathcal{U}\right)(x_1,\widehat{s}) + \sum_{k=0}^{+\infty} \left(-\sum_{p=2}^{n} s_p^{1+\alpha_p}\right)^{k+1} \left(I_{a_1^+}^{(1+\alpha_1)(k+2)}\mathcal{U}\right)(x_1,\widehat{s}) \\
&= \left(I_{a_1^+}^{1+\alpha_1}\mathcal{U}\right)(x_1,\widehat{s}) + \sum_{m=1}^{+\infty} \left(-\sum_{p=2}^{n} s_p^{1+\alpha_p}\right)^{m} \left(I_{a_1^+}^{(1+\alpha_1)(m+1)}\mathcal{U}\right)(x_1,\widehat{s}) \\
&= \sum_{m=0}^{+\infty} \left(-\sum_{p=2}^{n} s_p^{1+\alpha_p}\right)^{m} \left(I_{a_1^+}^{(1+\alpha_1)(m+1)}\mathcal{U}\right)(x_1,\widehat{s}).
\end{aligned} \tag{71}$$

Hence, from (71) and (31) (with $\lambda = 0$) we can rewrite (70) as

$$\begin{aligned}
\mathcal{V}(x_1,\widehat{s}) = &\sum_{m=0}^{+\infty} (-1)^m \frac{\left(\sum_{p=2}^{n} s_p^{1+\alpha_p}\right)^m}{\Gamma((1+\alpha_1)m+1)} (x_1-a_1)^{(1+\alpha_1)m}\,\mathfrak{G}_0(\widehat{s}) \\
&+ \sum_{m=0}^{+\infty} (-1)^m \frac{\left(\sum_{p=2}^{n} s_p^{1+\alpha_p}\right)^m}{\Gamma((1+\alpha_1)m+2)} (x_1-a_1)^{(1+\alpha_1)m+1}\,\mathfrak{G}_1(\widehat{s}) \\
&+ \sum_{m=0}^{+\infty} \left(-\sum_{p=2}^{n} s_p^{1+\alpha_p}\right)^m \left(I_{a_1^+}^{(1+\alpha_1)(m+1)}\mathcal{U}\right)(x_1,\widehat{s}).
\end{aligned} \tag{72}$$

It remains to invert the Laplace transform. Using (32) and after straightforward calculations, we obtain,

$$\begin{aligned}
\mathcal{V}(x_1,\widehat{s}) = &\sum_{k=0}^{\infty} (-1)^k \frac{(x_1-a_1)^{k(1+\alpha_1)}}{\Gamma((1+\alpha_1)k+1)} \left({}^{C}\widehat{\Delta}_{a^+}^{\alpha}\right)^k g_0(\widehat{x}) + \sum_{k=0}^{\infty} (-1)^k \frac{(x_1-a_1)^{(1+\alpha_1)k+1}}{\Gamma((1+\alpha_1)k+2)} \left({}^{C}\widehat{\Delta}_{a^+}^{\alpha}\right)^k g_1(\widehat{x}) \\
&+ \sum_{m=0}^{+\infty} \left(-\sum_{p=2}^{n} s_p^{1+\alpha_p}\right)^m I_{a_1^+}^{(1+\alpha_1)(m+1)} u(x)
\end{aligned} \tag{73}$$

which corresponds to our result.

■

Theorem 5.2 *Let* $q = \frac{2p}{2-(1-\alpha^*)p}$, $p \in \left]1, \frac{2}{1-\alpha^*}\right[$, *and* $\alpha^* = \min_{1\leq i\leq n}\{\alpha_i\}$. *The space* $L_q(\Omega)$ *admits the following direct decomposition*

$$L_q(\Omega) = L_q(\Omega) \cap \ker\left({}^{C}\mathcal{D}_{a^+}^{\alpha}\right) \oplus {}^{C}\mathcal{D}_{a^+}^{\alpha}\left(\overset{\circ}{W}{}_{a^+}^{\alpha,p}(\Omega)\right), \tag{74}$$

where $\overset{\circ}{W}{}^{\alpha,p}_{a^+}(\Omega)$ *is the space of functions* $g \in W^{\alpha,p}_{a^+}(\Omega)$ *such that* $\mathrm{tr}(g) = 0$*. Moreover, we can define the following projectors*

$$ {}^{C}P^{\alpha} : L_q(\Omega) \to L_q(\Omega) \cap \ker\left({}^{C}\mathcal{D}^{\alpha}_{a^+}\right), \qquad {}^{C}Q^{\alpha} : L_q(\Omega) \to {}^{C}\mathcal{D}^{\alpha}_{a^+}\left(\overset{\circ}{W}{}^{\alpha,p}_{a^+}(\Omega)\right). $$

Proof: By $(-\,{}^{C}\Delta^{\alpha}_{a^+})^{-1}_0$ we denote the unique operator solution for the problem (cf. Lemma 5.1)

$$ \begin{cases} -\,{}^{C}\Delta^{\alpha}_{a^+} f = u, & \text{in } \Omega \\ f = 0, & \text{on } \partial\Omega \end{cases} \tag{75} $$

which is given by (66) with $v(x) = f(x)$ and $g_0(\widehat{x}) = 0$.
As a first step we take a look at the intersection of the two spaces that appear in the decomposition. Let $f \in \left[L_q(\Omega) \cap \ker({}^{C}\mathcal{D}^{\alpha}_{a^+})\right] \cap {}^{C}\mathcal{D}^{\alpha}_{a^+}\left(\overset{\circ}{W}{}^{\alpha,p}_{a^+}(\Omega)\right)$. We directly see that ${}^{C}\mathcal{D}^{\alpha}_{a^+} f = 0$, in Ω. Moreover, since $f \in {}^{C}\mathcal{D}^{\alpha}_{a^+}\left(\overset{\circ}{W}{}^{\alpha,p}_{a^+}(\Omega)\right)$, there exists a function $g \in \overset{\circ}{W}{}^{\alpha,p}_{a^+}(\Omega)$ with ${}^{C}\mathcal{D}^{\alpha}_{a^+}\, g = f$ and ${}^{C}\Delta^{\alpha}_{a^+}\, g = 0$. From the uniqueness of $(-\,{}^{C}\Delta^{\alpha}_{a^+})^{-1}_0$ we obtain that $g = 0$. Consequently, $f = 0$ in Ω. Hence, the intersection of these subspaces only contains the zero function, which implies that the sum is direct.
Now, let $f \in L_q(\Omega)$ and f_2 such that

$$ f_2 := {}^{C}\mathcal{D}^{\alpha}_{a^+}\,(-\,{}^{C}\Delta^{\alpha}_{a^+})^{-1}_0\,{}^{C}\mathcal{D}^{\alpha}_{a^+} f \;\in\; {}^{C}\mathcal{D}^{\alpha}_{a^+}\left(\overset{\circ}{W}{}^{\alpha,p}_{a^+}(\Omega)\right). $$

Applying ${}^{C}\mathcal{D}^{\alpha}_{a^+}$ to the function $f_1 := f - f_2$, we get

$$ \begin{aligned} {}^{C}\mathcal{D}^{\alpha}_{a^+} f_1 &= {}^{C}\mathcal{D}^{\alpha}_{a^+} f - {}^{C}\mathcal{D}^{\alpha}_{a^+} f_2 \\ &= {}^{C}\mathcal{D}^{\alpha}_{a^+} f - {}^{C}\mathcal{D}^{\alpha}_{a^+}\,{}^{C}\mathcal{D}^{\alpha}_{a^+}\left(-\,{}^{C}\Delta^{\alpha}_{a^+}\right)^{-1}_0 {}^{C}\mathcal{D}^{\alpha}_{a^+} f \\ &= {}^{C}\mathcal{D}^{\alpha}_{a^+} f - \left(-\,{}^{C}\Delta^{\alpha}_{a^+}\right)\left(-\,{}^{C}\Delta^{\alpha}_{a^+}\right)^{-1}_0 {}^{C}\mathcal{D}^{\alpha}_{a^+} f \\ &= {}^{C}\mathcal{D}^{\alpha}_{a^+} f - {}^{C}\mathcal{D}^{\alpha}_{a^+} f = 0, \end{aligned} $$

i.e., $f_1 \in \ker\left({}^{C}\mathcal{D}^{\alpha}_{a^+}\right)$. Since $f \in L_q(\Omega)$ was arbitrarily chosen our decomposition is a direct decomposition of the space $L_q(\Omega)$.

■

Remark 5.3 *When* $\alpha = (1, \ldots, 1)$ *we have that* $q = p$ *and* $p \in]1, +\infty[$*. For the particular case of* $p = 2$ *the decomposition is orthogonal (see Theorem 3.75 in [17]).*

We end this section by presenting an immediate application of our results.

Theorem 5.4 *Let* $p \in \left]1, \frac{2}{1-\alpha^*}\right[$*, and* $\alpha^* = \min_{1\le i\le n}\{\alpha_i\}$*. Consider* $g \in W^{2+\alpha,p}_{a^+}(\Omega)$*. The unique solution of the problem*

$$ \begin{cases} {}^{C}\Delta^{\alpha}_{a^+} f = g, & \textit{in } \Omega \\ f = 0, & \textit{on } \partial\Omega \end{cases} $$

is given by $f = -\,{}^{C}T^{\alpha}\,{}^{C}Q^{\alpha}\,{}^{C}T^{\alpha} g$.

Proof: The proof is based on applying the properties of the operator ${}^{C}T^{\alpha}$ and of the projector ${}^{C}Q^{\alpha}$. Since ${}^{C}T^{\alpha}$ is the right inverse of ${}^{C}\mathcal{D}^{\alpha}_{a^+}$, we get

$$ {}^{C}\Delta^{\alpha}_{a^+} f = {}^{C}\mathcal{D}^{\alpha}_{a^+}\,{}^{C}\mathcal{D}^{\alpha}_{a^+}\,{}^{C}T^{\alpha}\,{}^{C}Q^{\alpha}\,{}^{C}T^{\alpha} g = {}^{C}\mathcal{D}^{\alpha}_{a^+}\,{}^{C}Q^{\alpha}\,{}^{C}T^{\alpha} g = {}^{C}\mathcal{D}^{\alpha}_{a^+}\,{}^{C}T^{\alpha} g = g. $$

■

6 Conclusion

In this work we presented a generalization of several results of the classical continuous Clifford function theory developed in [17] in the context of fractional Clifford analysis. Our results can be regarded as a starting point for future works. Due to the "double duality" indicated previously, some of the previous results admit alternative versions, for instance, for the operator ${}^{RL}\mathcal{D}^{\alpha}_{a^+}$. Moreover, it is desirable to obtain an explicit expression for the fundamental solutions finding appropriate functions g_0 and g_1. This can be done considering adequate series expansions in the neighbourhood of a. This will be subject for future work.

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